FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 43

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CONTENTS

1. FACTS WE'LL SOON KNOW ABOUT CURVES

We almost know enough to say a lot of interesting things about curves. There are a few more notions and facts that are very helpful, and I'll state them now as "black boxes" to take for granted. We'll prove everything in due course, and hopefully after seeing how useful they are, you'll be highly motivated to learn more.

For this topic, we will assume that all curves are projective, geometrically integral, nonsingular curves over a field k.

We will sometimes add the hypothesis that k is algebraically closed. Most people are happy with working over algebraically closed fields, and those people should ignore the adverb "geometrically" in the previous paragraph.

1.1. Differentials on curves.

Riemann surfaces (and complex manifolds more generally) support the notion of a *differential*, things which can be locally interpreted as $f(z)dz$, where z is a local parameter.

Similarly, there is a sheaf of differentials on a curve C, denoted Ω_C, which is an invertible sheaf. In general, a nonsingular k-variety X of dimension d will have a sheaf of differentials Ω_X that will be locally free of rank d. Its determinant is called the *canonical bundle* $\mathcal{K}_X$. In our case, $X = C$ is a curve, so $\mathcal{K}_C = \Omega_C$, and from here on in, we'll use $\mathcal{K}$ instead of Ω_C.

1.2. Serre duality.

The canonical bundle $\mathcal{K}$ is also an example of a *dualizing sheaf* because of its role in Serre duality. Serre duality states that (i) $H^1(C, \mathcal{K}) \cong k$. (ii) Further, for any coherent sheaf $\mathcal{F}$, the natural map

$$\boxed{H^0(C, \mathcal{F}) \otimes_k H^1(C, \mathcal{K} \otimes \mathcal{F}^\vee) \to H^1(C, \mathcal{K}) \cong k}$$

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is a perfect pairing. (This is our first black box! — remember this, as we will use it repeatedly!) Thus in particular, $h^0(C, \mathcal{F}) = h^1(C, \mathcal{K} \otimes \mathcal{F}^\vee)$. Recall we defined the arithmetic genus of a curve to be $h^1(C, \mathcal{O}_C)$. Hence $h^0(C, \mathcal{K}) = g$ as well: there is a g-dimensional family of differentials.

1.3. *Proposition.* — $\deg \mathcal{K} = 2g - 2$.

Proof. Recall that Riemann-Roch for a invertible sheaf $\mathcal{L}$ states that

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} - g + 1.$$

Applying this to $\mathcal{L} = \mathcal{K}$, we get

$$\deg \mathcal{K} = h^0(C, \mathcal{K}) - h^1(C, \mathcal{K}) + g - 1 = h^1(C, \mathcal{O}) - h^0(C, \mathcal{O}) + g - 1 = g - 1 + g - 1 = 2g - 2.$$

□

1.4. *Example.* If $C = \mathbb{P}^1$, then the above Proposition implies $\mathcal{K} \cong \mathcal{O}(-2)$. Here is a heuristic which will later be made precise. On the affine open subset $x_0 \neq 0$, given by $\operatorname{Spec} k[x_{1/0}]$, we expect $dx_{1/0}$ to be a differential, which has no poles or zeros. Let's analyze this as a differential on an open subset of the other affine open subset, $\operatorname{Spec} k[x_{0/1}]$, where $x_{0/1} = 1/x_{1/0}$. If differentials behave the way we are used to, then $dx_{1/0} = -(1/x_{0/1}^2) dx_{0/1}$. Thus we expect that the rational differential $dx_{1/0}$ on $\mathbb{P}^1$ to have no zeros, and a pole at order 2 "at ∞", so the line bundle of differentials must be isomorphic to $\mathcal{O}(-2)$.

Part (i) of Serre duality certainly holds: $h^1(\mathbb{P}^1, \mathcal{O}(-2)) = 1$. Moreover, we also have a natural perfect pairing

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) \times H^1(\mathbb{P}^1, \mathcal{O}(-2-n)) \to k.$$

If $n < 0$, both factors on the left are 0, so we assume $n > 0$. Then $H^0(\mathbb{P}^1, \mathcal{O}(n))$ corresponds to homogeneous degree n polynomials in x and y, and $H^1(\mathbb{P}^1, \mathcal{O}(-2-n))$ corresponds to homogeneous degree $-2-n$ Laurent polynomials in x and y so that the degrees of x and y are both at most $n-1$. You can quickly check that the dimension of both vector spaces are $n+1$. The pairing is given as follows: multiply the polynomial by the Laurent polynomial, to obtain a Laurent polynomial of degree -2. Read off the co-efficient of $x^{-1}y^{-1}$.

1.5. The Riemann-Hurwitz formula.

Differentials pull back: any surjective morphism of curves $f : C \to C'$ induces a natural map $f^*\Omega_{C'} \to \Omega_C$.

Suppose $f : C \to C'$ is a dominant morphism. Then it turns out $f^*\Omega_{C'} \hookrightarrow \Omega_C$ is an inclusion of invertible sheaves. (This is a case when inclusions of invertible sheaves does not mean what people normally mean by inclusion of line bundles, which are always isomorphisms.) The fact that this is injective arises from the fact that Ω_C is a line bundle, and hence torsion-free, and thus has no non-zero torsion subsheaves. But $f^*\Omega_{C'} \to \Omega_C$ is non-zero at the generic point, so the kernel is necessarily torsion.

Its cokernel is supported in dimension 0:

$$0 \to f^*\Omega_{C'} \to \Omega_C \to [\text{dimension } 0] \to 0.$$

The divisor R corresponding to those points (with multiplicity), is called the **ramification divisor**.

By an Exercise from the last couple of classes, the degree of the pullback of an invertible sheaf is the degree of the map times the degree of the original invertible sheaf. Thus if d is the degree of the cover, $\deg \Omega_C = d \deg \Omega_{C'} + \deg R$. Hence if $C \to C'$ is a degree d cover of curves, then

$$\boxed{2g_c - 2 = d(2g_{C'} - 2) + \deg R}$$

This is our second black box. Remember it!

Let's now figure out how to measure $\deg R$. We can study this in local coordinates. We don't have the technology to describe this precisely yet, so we'll stick to the case where $\operatorname{char} k = 0$ and k is algebraically closed whenever we use the Riemann-Hurwitz formula, until we formally prove things. Heuristically, if the map at $q \in C'$ looks like $u \mapsto u^n = t$, then $dt \mapsto d(u^n) = nu^{n-1}du$, so dt when pulled back vanishes to order $n-1$. Thus branching of this sort $u \mapsto u^n$ contributes $n-1$ to the ramification divisor. (More correctly, we should look at the map of Spec's of discrete valuation rings, and then u is a uniformizer for the stalk at q, and t is a uniformizer for the stalk at $f(q)$, and t is actually a unit times u^n. But the same argument works.)

1.A. EASY BUT CRUCIAL EXERCISE. Suppose $C \to C'$ is a degree d map of nonsingular projective curves over k ($\operatorname{char} k = 0$ and $k = \overline{k}$), and the closed points $p \in C'$ has e preimages (set-theoretically). Show that the amount of ramification above p (the degree of the part of the ramification divisor supported in the preimage of p) is $d - e$.

Here are some applications.

1.B. EXERCISE. Show that there is no nonconstant map from a genus 2 curve to a genus 3 curve. (Hint: $\deg R \geq 0$.)

1.6. *Example: Hyperelliptic curves..* Hyperelliptic curves are curves that are double covers of $\mathbb{P}^1_k$. If they are genus g, then they are branched over $2g+2$ points, as each ramification can happen to order only 1. (Warning: we are in characteristic 0!) You may already have heard about genus 1 complex curves double covering $\mathbb{P}^1$, branched over 4 points.

1.7. *Example.* For any map, the degree of R is even: any cover of a curve must be branched over an even number of points (counted with multiplicity).

1.8. *Example.* The only connected unbranched cover of $\mathbb{P}^1_k$ is the isomorphism. Reason: if $\deg R = 0$, then we have $2 - 2g_C = 2d$ with $d \geq 1$ and $g_c \geq 0$, from which $d = 1$ and $g_C = 0$.

1.9. *Example: Lüroth's theorem..* Suppose $g(C) = 0$. Then from the Riemann-Hurwitz formula, $g(C') = 0$. (Otherwise, if $g_{C'}$ were at least 1, then the right side of the Riemann-Hurwitz formula would be non-negative, and thus couldn't be -2, which is the left side. This has a non-obvious algebraic consequence, by our identification of covers of curves with field extensions. All subfields of $k(x)$ containing k are of the form $k(y)$ where $y = f(x)$. (It turns out that the hypotheses that $\operatorname{char} k = 0$ and $k = \overline{k}$ are not necessary; we'll remove them in due course.)

1.10. A criterion for when a morphism is a closed immersion.

The third fact we need is a criterion for when something is a closed immersion. This won't need to be a black box — we'll be able to prove it. To help set it up, let's recall some facts about closed immersions. Suppose $f : X \to Y$ is a closed immersion. Then f is projective, and it is injective on points. This is not enough to ensure that it is a closed immersion, as the example of the normalization of the cusp shows (Figure 1). Another example is the *Frobenius morphism* from $\mathbb{A}^1$ to $\mathbb{A}^1$, given by $k[t] \to k[u]$, $u \to t^p$, where k has characteristic p.

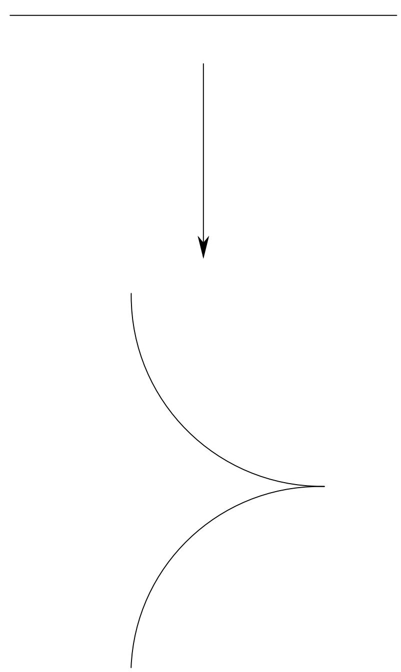

FIGURE 1. Projective morphisms that are injective on points need not be closed immersions

The additional information you need is that the tangent map is an isomorphism at all closed points.

1.C. EXERCISE. Show that in the two examples described above (the normalization of a cusp and the Frobenius morphism), the tangent map is *not* an isomorphism at all closed point.

1.11. *Theorem. — Suppose* $k = \overline{k}$, *and* $f : X \to Y$ *is a projective morphism of finite-type* k*-schemes that is injective on closed points and injective on tangent vectors at closed points. Then* f *is a closed immersion.*

(Remark: this is the definition of an *unramified* map in this situation. We will later define this in more generality.)

The example $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ shows that we need the hypothesis that k is algebraically closed. For those of you who are allergic to algebraically closed fields: still pay attention, as we'll use this to prove things about curves over k where k is *not* necessarily algebraically closed.

We need the hypothesis of projective morphism, as shown by the example of Figure 2. It is the normalization of the node, except we erase one of the preimages of the node. We map $\mathbb{A}^1$ to the plane, so that its image is a curve with one node. We then consider the morphism we get by discarding one of the preimages of the node. Then this morphism is an injection on points, and is also injective on tangent vectors, but it is not a closed immersion. (In the world of differential geometry, this fails to be an embedding because the map doesn't give a homeomorphism onto its image.)

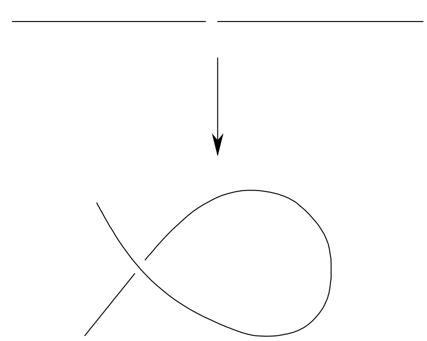

FIGURE 2. We need the projective hypothesis in Theorem 1.11

Suppose $f(p) = q$, where p and q are closed points. We will use the hypothesis that X and Y are finite type k-schemes where k is algebraically closed at only one point of the argument: that the map induces an isomorphism of residue fields at p and q.

This is the hardest result of today. We will kill the problem in old-school French style: death by a thousand cuts.

Proof. The property of being a closed immersion is local on the base, so we may assume that Y is affine, say $\operatorname{Spec} B$.

I next claim that f has finite fibers, not just finite fibers above closed points: the fiber dimension for projective morphisms is upper-semicontinuous (an earlier exercise), so the locus where the fiber dimension is at least 1 is a closed subset, so if it is non-empty, it must contain a closed point of Y. Thus the fiber over any point is a dimension 0 finite type scheme over that point, hence a finite set.

Hence f is a projective morphism with finite fibers, thus finite (an earlier corollary).

So far this argument is a straightforward sequence of reduction steps and facts we know well. But things now start to get weird.

Thus X is affine too, say $\operatorname{Spec} A$, and f corresponds to a ring morphism $B \to A$. We wish to show that this is a surjection of rings, or (equivalently) of B-modules. We will show that for any maximal ideal $\mathfrak{n}$ of B, $B_{\mathfrak{n}} \to A_{\mathfrak{n}}$ is a surjection of $B_{\mathfrak{n}}$-modules. (This will show that $B \to A$ is a surjection. Here is why: if K is the cokernel, so $B \to A \to K \to 0$, then we wish to show that $K = 0$. Now A is a finitely generated B-module, so K is as well, being a homomorphic image of A. Thus $\operatorname{Supp} K$ is a closed set. If $K \neq 0$, then $\operatorname{Supp} K$ is non-empty, and hence contains a closed point $[\mathfrak{n}]$. Then $K_{\mathfrak{n}} \neq 0$, so from the exact sequence $B_{\mathfrak{n}} \to A_{\mathfrak{n}} \to K_{\mathfrak{n}} \to 0$, $B_{\mathfrak{n}} \to A_{\mathfrak{n}}$ is not a surjection.)

If $A_{\mathfrak{n}} = 0$, then clearly $B_{\mathfrak{n}}$ surjects onto $A_{\mathfrak{n}}$, so assume otherwise. I claim that $A_{\mathfrak{n}} = A \otimes_B B_{\mathfrak{n}}$ is a local ring. Proof: $\operatorname{Spec} A_{\mathfrak{n}} \to \operatorname{Spec} B_{\mathfrak{n}}$ is a finite morphism (as it is obtained by base change from $\operatorname{Spec} A \to \operatorname{Spec} B$), so we can use the going-up theorem. $A_{\mathfrak{n}} \neq 0$, so $A_{\mathfrak{n}}$ *has* a prime ideal. Any point p of $\operatorname{Spec} A_{\mathfrak{n}}$ maps to some point of $\operatorname{Spec} B_{\mathfrak{n}}$, which has $[\mathfrak{n}]$ in its closure. Thus there is a point q in the closure of p that maps to $[\mathfrak{n}]$. But there is only one point of $\operatorname{Spec} A_{\mathfrak{n}}$ mapping to $[\mathfrak{n}]$, which we denote $[\mathfrak{m}]$. Thus we have shown that $\mathfrak{m}$ contains all other prime ideals of $\operatorname{Spec} A_{\mathfrak{n}}$, so $A_{\mathfrak{n}}$ is a local ring.

Here things get weirder still. We apply Nakayama, using two *different* local rings.

Injectivity of tangent vectors *means* surjectivity of cotangent vectors, i.e. $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is a surjection, i.e. $\mathfrak{n} \to \mathfrak{m}/\mathfrak{m}^2$ is a surjection. I claim that $\mathfrak{n}A_{\mathfrak{n}} \to \mathfrak{m}A_{\mathfrak{n}}$ is an isomorphism. Reason: Using Nakayama's lemma for the local ring $A_{\mathfrak{n}}$ and the $A_{\mathfrak{n}}$-*module* $\mathfrak{m}A_{\mathfrak{n}}$, we conclude that $\mathfrak{n}A_{\mathfrak{n}} = \mathfrak{m}A_{\mathfrak{n}}$.

Next apply Nakayama's Lemma to the $B_{\mathfrak{n}}$-*module* $A_{\mathfrak{n}}$. The element $1 \in A_{\mathfrak{n}}$ gives a generator for $A_{\mathfrak{n}}/\mathfrak{n}A_{\mathfrak{n}} = A_{\mathfrak{n}}/\mathfrak{m}A_{\mathfrak{n}}$, which equals $B_{\mathfrak{n}}/\mathfrak{n}B_{\mathfrak{n}}$ (as both equal k), so we conclude that 1 also generates $A_{\mathfrak{n}}$ as a $B_{\mathfrak{n}}$-module as desired. □

1.D. EXERCISE. Use this to show that the dth Veronese morphism from $\mathbb{P}^n_k$, corresponding to the complete linear series $|\mathcal{O}_{\mathbb{P}^n_k}(d)|$, is a closed immersion. Do the same for the Segre morphism from $\mathbb{P}^m_k \times_{\operatorname{Spec} k} \mathbb{P}^n_k$. (This is just for practice for using this criterion. This is a weaker result than we had before; we've earlier checked this over an arbitrary base ring, and we are now checking it only over algebraically closed fields.)

Although Theorem 1.11 requires k to be algebraically closed, the following exercise will enable us to use it for general k.

1.E. EXERCISE. Suppose $f : X \to Y$ is a morphism over k that is affine. Show that f is a closed immersion if and only if $f \times_k \overline{k} : X \times_k \overline{k} \to Y \times_k \overline{k}$ is. (The affine hypothesis is certainly not necessary for this result, but it makes the proof easier, and this is the situation in which we will most need it.)

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