How a Little Bit goes a Long Way: Predicative Foundations of Analysis

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Contents

I. Introduction

The following typescript notes serve a double purpose. First of all, they substantiate in detail assertions that I made in several publications, including Feferman (1985, 1988, 1992, and 2005) that most of classical analysis and substantial portions of modern analysis can be developed on the basis of a system conservative over Peano Arithmetic (PA). The informal development along these lines was initiated by Hermann Weyl in his groundbreaking monograph *Das Kontinuum* (1918), and is called *predicative analysis.* Roughly speaking, *predicative mathematics* in general is that part of mathematics that is implicit in accepting the natural numbers as the only completed infinite totality. As explained in the survey paper Feferman (2005), the philosophical ideas for that go back to Henri Poincaré and Bertrand Russell in the early 20th century. The word 'predicative' has also been applied to other developments, so the present approach is sometimes distinguished as *predicativity, given the natural numbers.* According to the logical analysis of that notion in general by Feferman (1964) and (independently) by Schütte (1965), its limits go far beyond PA in strength (see the end of this introduction for some metatheoretical and mathematical descriptions of that). Nevertheless, there is special interest in seeing how much can be done in arithmetic as the initial predicative system because of a conjecture that I made at the end of Feferman (1987), namely that *all scientifically applicable mathematics can be formalized in a theory conservative over PA*. A verification of that conjecture would thus show that scientifically applicable mathematics does not require the assumption of impredicative set theory or any uncountable cardinals for its eventual justification. See Feferman (1992, 2005) for further discussion of the philosophical significance of this conjecture.

The second purpose of these notes is to solve an expository problem that has arisen in the work in progress on a book, *Foundations of Explicit Mathematics*, which is being written in collaboration with Gerhard Jäger and Thomas Strahm with the assistance of Ulrik Buchholtz. As explained in my draft introduction to that work:

> Explicit Mathematics is a flexible unified framework for the systematic logical study of those parts of higher mathematics in which proofs of existence guarantee the computability or definability by specified means of what is thereby

> demonstrated to exist. … [T]he main parts of mathematics covered by the Explicit Mathematics framework are referred to as *constructive*, *predicative*, and *descriptive* …; each was originally pursued on philosophical grounds that—whatever their merits—have been thought too confining to support mathematical practice and its scientific applications. To the contrary, what the present work shows through the logical analysis provided by our framework is that in gaining the uniform explicitness of solutions little is lost in terms of both the workability and applicability of these approaches, despite their philosophical and methodological restrictions.

The initial formulation of systems of explicit mathematics was made in Feferman (1975) and was continued a few years later in Feferman (1979). The subsequent development of the subject has been considerable and has proved to be adaptable to a variety of other contexts than the ones just indicated, ranging from theories of feasible computation and finitist mathematics to large cardinals in set theory. The aim of the book in progress is to provide a substantial introduction to the subject including a full presentation of the main formal systems involved, their models, and the evaluation of their proof-theoretic strengths.

It is also intended to devote a part of the book on explicit mathematics to explaining via basic notions and some typical arguments how one goes about formalizing various parts of constructive, predicative and descriptive mathematics in appropriate ones of these systems. In the case of modern (non-Brouwerian) constructive analysis, the task is made easy by direct reference to the work of Bishop (1967). And in the case of descriptive set theory, there are a number of sources in the literature that can easily be followed, such as Moschovakis (1980). However, in the case of predicative analysis (beyond that treated by Weyl) there are only three possible references, namely Grzegorczyk (1955), Lorenzen (1965) and Simpson (1998), none of which succeeds in directly meeting the present purposes (as will be explained below) though Simpson's work comes by far the closest. Part II of these notes *do* meet the present purposes in full and will serve as the needed reference for the planned chapter on predicative analysis in the book on explicit mathematics.

First, some background. These notes were prepared in the period 1977-1981, but never published. They were taken from a group of notes originally consisting of five parts I-V; the notes following this introduction constitute Part II. The notes I-V as a whole formed a draft for a book that was intended to elaborate the material of the article, "Theories of finite type related to mathematical practice" (Feferman 1977) in the *Handbook of Mathematical Logic* (Barwise 1977).[1] The aim of that article was to provide a theoretical framework for the natural development of constructive, predicative and descriptive analysis that could be treated proof-theoretically to extract information as to strength, explicit definability and conservation results. It was noted there that the bulk of those parts of practice can be carried out within the finite type structure over the set N of natural numbers, and indeed within type level three, counting N as being at type level 0, N^N and the real numbers at type level 1, functions of real numbers at type level 2, and functional operators at type level 3. As it happened, though, the proof theory employed in the 1977 article works for all finite types via Gödel's method of functional interpretation and some of its extensions. The finite type structure in question can be conceived to consist of the *types* (aka *classes*) S, T,… generated from N by closure under Cartesian product $S \times T$ and Cartesian power T^S, written alternatively as $(S \rightarrow T)$, though product can be replaced by power via "Currying." By contrast, the bulk of the literature on the proof theory of analysis has been devoted to the study of subsystems of second-order arithmetic, with the variables of type level 1 taken to range over the subsets of N, in which the real numbers can be directly represented, but where the needed functions and functionals of such are represented only indirectly by certain kinds of coding. At the same time, even the finite type structure is an oversimplification as a framework for the direct representation of practice. First of all, the types should not be treated as *fixed objects*, but rather as *variable objects* in order to talk about *arbitrary spaces* (e.g. metric, linear, Hilbert, Banach, etc.) of the sort ubiquitous in modern analysis. Moreover, for the natural representation of those spaces one must also have closure under *subtypes* of the form $\{x \in S \mid \phi(x)\}$ for ϕ a formula. What was done in Part I of the original notes (I-V)

[1] There is a kind of circularity here: In fn 2 of Feferman (1977), it is said that the plan for that article is derived from a book by me entitled *Explicit Content of Actual Mathematical Analysis* slated to appear in 1978, a promissory note that was not fulfilled.

was to provide a formal framework for a system meeting these requirements. Instead of reproducing that rather long part here, it is sufficient for our purposes as background to Part II to sketch its setup in the following; an alternative source for more details is the article, "A theory of variable types" (Feferman 1985).[2]

In that article, VT abbreviates "Variable Types", while in the notes below I have instead used VFT for "Variable Finite Types", to emphasize the relation with the theory of fixed finite types over N described above. We start with a base system VFT_0 whose language is given by the simultaneous inductive generation of *individual terms*, *type terms*, and *formulas*, as well as the relation, t *is of type* T, as follows:

1. Individual terms (s, t, u,…)

a) With each type term T is associated an infinite list of individual variables x^T, y^T, z^T, … of type T.

b) If s is of type S and t is of type T, then (s, t) is of type S × T.

c) If u is of type S × T then $p_1(u)$ is of type S and $p_2(u)$ is of type T.

d) If s is of type S and t is of type S → T then ts [or t(s)] is of type T.

e) If t is of type T, then $\lambda x^S.t$ is of type S → T.

2. Type terms (S, T,…)

a) Each type variable X, Y, Z, … is a type term.

b) If S, T are type terms and ϕ is a formula, then S × T, S → T, and $\{x^S \mid \phi\}$ are type terms.

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[2] A scan of that article is available on my home page at `http://math.stanford.edu/~feferman/papers/TheoryVarTypes.pdf`; unfortunately it is not fully readable, but the essentials can be gleaned. The material of that article itself was presented at the Fifth Latin American Symposium on Mathematical Logic, held in Bogotá, Colombia in July 1981, the proceedings of which did not appear until 1985.

3. Formulas (ϕ, ψ,…)

a) Each equation $t_1 = t_2$ between individual terms of arbitrary type [not necessarily the same!] is a formula.

b) If ϕ, ψ are formulas then so also are $\neg\phi$ and $\phi \rightarrow \psi$.

c) If ϕ is a formula and S is a type term, then $\forall x^S\ \phi$ is a formula.

The logic of VFT_0 is that of the many-sorted classical predicate calculus. The operations on formulas given by $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \leftrightarrow \psi$ and $\exists x^S\ \phi$ are then defined classically as usual. Also we write $t \in T$ for $\exists x^T\ (t = x)$ ('x' not in t), and then $S \subseteq T$ is defined in the standard way; S = T is defined to hold when $S \subseteq T$ and $T \subseteq S$. We do not assume extensionality for either functions or types.

VFT_0 has three general axioms, I-III. Axiom I is for typed λ conversion as usual. Axiom II is for pairing and projections, i.e. it tells us that for each X, Y and x in X, y in Y, $p_1(x,y) = x$ and $p_2(x,y) = y$, and that for each z in $X \times Y$, $z = (p_1(z), p_2(z))$. Finally, Axiom III is the Separation Axiom, according to which for each X, $\{x \in X \mid \phi(x,\ldots\} \subseteq X$ and for each y in X,

$$y \in \{x \in X \mid \phi(x,\ldots)\} \leftrightarrow \phi(y,\ldots).$$

The system VFT is an extension of VFT_0 by a language and axioms for the natural numbers. We adjoin the constant type symbol N, individual constants 0 and sc of type N and $N \rightarrow N$, resp., and individual recursion terms r_T of type $((N \times T \rightarrow T) \times T) \rightarrow (N \rightarrow T)$ for each type T. We use the letters 'n', 'm', 'p',… to range as variables over N and 'f', 'g', 'h',… to range as variables over various function types $S \rightarrow T$. VFT adds the following Axioms IV-VI. Axiom IV is the usual one for 0 and sc; we also write n' for sc(n). Axiom V is Induction, in the form

$$0 \in X \wedge \forall n\ (n \in X \rightarrow n' \in X) \rightarrow N \subseteq X.$$

Finally, Axiom VI is for Recursion on N into an arbitrary type T. This tells us that if f is of type $N \times T \rightarrow T$ and $a \in T$ and $g = r_T(f, a)$ then

$$g(0) = a \wedge \forall n[g(n') = f(n, g(n))].$$

The system VFT has a model in which the types are all the arithmetically definable subsets of the natural numbers N. In particular, $S \to T$ for any two such types S and T is interpreted to be the set of all indices e of partial recursive functions whose domain includes S and which map S into T. VFT also has a classical model in the cumulative hierarchy up to level ω over N considered as a set of urelements. It is remarked several times in Part II of the notes below that VFT suffices for the formalization of *constructive analysis* in the sense of Bishop (1967) [and, thus, equally well in Bishop and Bridges (1985)].

Theorem 1. VFT is a conservative extension of PA.

This may be established by a quick model-theoretic proof as follows. Let M be any model of PA; then M can be expanded to a model M* of VFT by taking the types to range over all first-order definable subsets of M. Let S, T be any types of M*. Using standard pairing and projection operations in M, $S \times T$ is defined as usual, and $S \to T$ is defined to consist of all indices z in M such that for each x in S, $\{z\}(x)$ is in T. Finally, each formula ϕ is equivalent to a formula that is first-order definable over M, so $\{x \in S \mid \phi(x)\}$ is also a type in M*. (The theorem can also be established by a proof-theoretic argument.)

We now turn to a system obtained from VFT—in part by an expansion and in part by a restriction—that is suitable for the formalization of *predicative analysis* and that is also conservative over PA. The expansion is given by adjunction of a constant μ for the *unbounded minimum operator* on N; it is of type $(N \to N) \to N$ and comes with the following additional axiom:

(μ) $$f \in (N \to N) \wedge f(n) = 0 \to f(\mu f) = 0 \wedge \mu f \leq n.$$

It may be seen that the system VFT + (μ) is stronger than PA since we can prove by the induction axiom of VFT transfinite induction up through ε_0 (at least) for elementary arithmetical properties with function parameters. To cut down its strength to PA, we now restrict the axioms of induction and recursion in VFT as follows. By a *subset x of N*, we mean a subtype of N that has a characteristic function. We can simply identify such x with an element of $(N \to N)$ and write $n \in x$ for $x(n) = 0$. Then the *Set Induction Axiom* tells us that for each subset x of N,

$$0 \in x \wedge \forall n[n \in x \to n' \in x] \to \forall n(n \in x).$$

This is equivalent to the statement:

$$f, g \in N \to N \wedge f(0) = g(0) \wedge \forall n[f(n) = g(n) \to f(n') = g(n')] \to \forall n[f(n) = g(n)].$$

The second restriction made is to take r_N as the only recursion operator. By Res-VFT is meant the system obtained from VFT by replacing the Induction Axiom by the Set Induction Axiom and the Recursion Axiom VI by its special case for r_N. (*NB*: In the notes below, Res-VFT is also written as VFT followed by the restriction sign.)

Theorem 2. Res-VFT + (μ) is a conservative extension of PA.

A proof of this is sketched in Feferman (1985). A proof of a stronger result is given in terms of certain systems of explicit mathematics in Feferman and Jäger (1996) whose language and axioms are simpler than those given by the VFT systems. In particular, it has a universal type V and all individual variables range over V; the reader is referred to that article for all details of language and axioms. A base system for our work is called *Elementary Explicit Type Theory* and is denoted by EET. The Axiom of Induction of VFT is called *Type Induction* when added to EET, and is denoted there by (T-I_N). Similarly the Set Induction Axiom is denoted by (S-I_N). Finally, the addition of the (μ) axiom to EET is denoted by EET(μ). The first of the following theorems connecting the two approaches is quite direct.

Theorem 3.

(i) VFT is interpretable in EET + (T-I_N).

(ii) Res-VFT + (μ) is interpretable in EET(μ) + (S-I_N).

Theorem 4. (Feferman and Jäger 1996)

(i) EET + (T-I_N) is proof-theoretically equivalent to PA and is a conservative extension of it.

(ii) EET(μ) + (S-I_N) is proof-theoretically equivalent to PA and is a conservative extension of it.

Thus these results serve to supersede the arguments of Feferman (1985). Like VFT, the system EET + (T-I_N) has both a recursion theoretic model and a classical model. It will be shown in the forthcoming book on explicit mathematics how to carry out within it typical notions and arguments of Bishop style constructive analysis as given in Bishop and Bridges (1985).[3] What Part II of these notes provide is to show in detail how all of 19th c. classical analysis and much of 20th c. analysis can be carried out in a generally straightforward way within Res-VFT + (μ), hence in EET(μ) + (S-I_N).

More specifically, in the case of 19th c. analysis, systematic use is made of Cauchy completeness rather than the impredicative l.u.b. principle, and sequential compactness is used in place of the Heine-Borel theorem. Then for 20th c. analysis, Lebesgue measurable sets and functions are treated directly without first going through the impredicative operation of outer measure; the existence of non-measurable sets cannot be proved in our system. Moving on to functional analysis, again the "positive" theory can be developed, at least for separable Banach and Hilbert spaces, and can be applied to various L_p spaces as principal examples. Among the general results that are obtained are usable forms of the Riesz Representation Theorem, the Hahn-Banach Theorem, the Uniform Boundedness Theorem, and the Open Mapping Theorem. The notes conclude with the spectral theory for compact self-adjoint operators on a separable Hilbert space. It is in this way that the assertions in the publications referred to at the beginning are hereby substantiated.

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[3] Variant designations for the systems of Explicit Mathematics involved will be used there.

We can now say something about how this work compares with the developments of predicative analysis in Grzegorczyk (1955), Lorenzen (1965) and Simpson (1988). Grzegorczyk's aim was to give a precise model for Weyl (1918) in terms of the notions of elementarily (i.e. arithmetically) definable real numbers, real functions and sets of real numbers. It does not go beyond 19^{th} c. analysis and no attention is paid to proof-theoretical strength. Lorenzen, in his book, also conceives of his work as an extension of Weyl (1918), but only to differential geometry. Moreover, though his work begins with a sketch of a foundational approach, there are no proof-theoretic results. Simpson's work on the other hand is quite different from both of these. It is an exposition of the work in the Reverse Mathematics program initiated by Harvey Friedman that centers on five *subsystems of second order analysis*: RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0. Each of these beyond the first is given by a single second-order axiom, in addition to the Induction Axiom as in Axiom V of VFT. In contrast to our work, which permits the free representation of practice in the full variable finite type structure over N, all mathematical notions considered by Simpson are represented in the second-order language by means of considerable coding. The main aim of the Reverse Mathematics program is to show that for a substantial part of practice, if a mathematical theorem follows from a suitable one of the basic axioms then it is equivalent to it, i.e. the implication can be reversed. For comparison with our work, much of predicative analysis falls under these kinds of results obtained for WKL_0 and ACA_0, that are of proof-theoretical strength PRA and PA respectively. Thus, on the one hand Simpson's results are proof-theoretically stronger than ours, since the strength of various individual theorems of analysis is sharply determined. On the other hand, the exposition for the work in WKL_0 and ACA_0 is not easily read as a systematic development of predicative analysis, as it is in our notes. Still, the Simpson book is recommended as a rich resource of other interesting results that could be incorporated into our development.

In conclusion, something must be said about the outer limits of predicativity both from a metatheoretical and from a mathematical point of view. The reference Feferman (2005) gives a general account of both, and the articles Friedman (2002) and Simpson (2002) concern specific results dealing with the latter. The reader is directed to these for further references not given below. Initially, in the 1950s, predicativity was studied as a part of the theory of definable sets of natural numbers, where quantification over N is accepted to be the basic definite logical operation. With this in mind, Kleene introduced the collection HYP of hyperarithmetical sets in terms of what one obtains by iterating the numerical quantification ("jump") operation through the constructive ordinals, i.e. through those ordinals with a recursive order type. Write $\omega_1^{(rec)}$ for the least non-recursive ordinal. Using H_α to denote the set obtained at the αth level in that way (taking effective joins at limit ordinals), Kleene defined HYP to be the collection of all sets recursive in some H_α for $\alpha < \omega_1^{(rec)}$. An alternative description of HYP may be obtained using the ramified hierarchy R_α of sets of natural numbers. That is defined for $\alpha < \omega_1^{(rec)}$ by taking R_α to consist of the sets given by definitions in the ramified language of 2nd order arithmetic in which the 2nd order variables are each restricted to range over some R_β for some $\beta < \alpha$. This corresponds to the requirement of predicative definability that one only deals with those classes of sets defined in terms of previously accepted classes; Kleene showed that the union of the R_α for $\alpha < \omega_1^{(rec)}$ equals HYP. Finally, Spector showed that *bootstrapping* through the predicatively definable ordinals does not take one beyond HYP, since every HYP definable well-ordering of N is of the same order type as a recursive well-ordering. If HYP is accepted as an upper bound for the *predicatively definable sets*, that can be used to show that certain theorems of analysis are *impredicative*. One example is the existence of non-Lebesgue measurable sets of reals, since every HYP set of reals is measurable. Also, Kreisel showed that the Cantor-Bendixson theorem, according to which every closed set is the union of a perfect set and a countable (scattered) set fails in HYP.

Moving on, Kreisel proposed that for a proper analysis of predicativity—that is, of what notions and principles concerning them one ought to accept if one has accepted the natural numbers—it would be more appropriate to deal with *predicative provability* rather than predicative definability. Kreisel's suggestion was that this should be done in terms of an *autonomous transfinite progression of systems* RA_α *of ramified analysis* where the autonomy (or boot-strap) restriction is that one ascends only to those levels α for which some recursive relation of order type α has been proved to be a well-ordering in RA_β for some $\beta < \alpha$. It was independently established in Feferman (1964) and Schütte

(1965) that the *least impredicative ordinal*, i.e. the limit of the predicatively provable ordinals in this sense, is the least fixed point α of $\chi_\alpha(0) = \alpha$, where the χ_α form the first Veblen hierarchy of critical functions of ordinals for all ordinals α. The least impredicative ordinal is denoted Γ_0 and we have $\Gamma_0 < \omega_1^{(rec)}$. Thus, even if a mathematical statement holds in HYP it will be impredicative if, for example, it proves the consistency of the union of the RA_α for $\alpha < \Gamma_0$. The first such example was provided by Friedman who obtained a finite combinatorial form of Kruskal's Theorem of this kind; Kruskal's Theorem itself—an infinitary statement—implies the well-foundedness of the standard ordering of order type Γ_0 (cf. Friedman (2002) for the back references).

The investigation via formal theories of which parts of analysis are predicatively justified is best pursued via unramified systems T, since ramification is an artificial restriction on the language of analysis in practice. Feferman (1964) initiated the study of *predicatively reducible unramified systems*, i.e. those systems T that are proof-theoretically reducible to some RA_α for $\alpha < \Gamma_0$. Of course the system ACA_0, which is another form of the lowest level in the hierarchy of ramified theories, is trivially predicatively reducible. At the opposite end, there are a number of interesting systems to mention that are of the same proof-theoretical strength as the union of the RA_α for $\alpha < \Gamma_0$. First of all, we have the system Σ^1_1-DC + BR; the proof-theoretical equivalence in this case was first established in Feferman (1978) and later as a special case of a more general statement in Feferman and Jäger (1983). In the latter publication, another system of this type is formulated as the autonomous iteration of the Π^0_1 comprehension axiom. Finally, Friedman, McAloon and Simpson (1982) showed that the system ATR_0 is also of the same proof-theoretical strength as full predicative analysis. Since that is given by a single axiom over RCA_0, it follows that results in analysis and other parts of mathematics that are provably equivalent to ATR_0 are impredicative. Simpson (2002, 2010) gives a number of examples of theorems from descriptive set theory that are equivalent to ATR_0, such as that every uncountable closed (or analytic) set contains a perfect subset. Also, ATR_0 is equivalent to comparability of countable well-orderings. In addition, Friedman, McAloon and Simpson (1982) give a mathematically natural finite combinatorial theorem that is equivalent to the (1-) consistency of ATR_0, and hence not provable from it.

To conclude, something should be said about the brief final section 5 of Part II of these notes. That section concerns some uses of the axiom $Proj_1$ of *type 1 projection* in an extension of the VFT framework. $Proj_1$ is an axiom for quantification over $N \rightarrow N$ that gives a system of strength full 2nd order analysis in the presence of the (μ) axiom. The sentence $Proj_1$ says that if b is any *subset* of $N \times (N \rightarrow N)$ then there is a *subset* a of N such that

$$\forall n[\, n \in a \leftrightarrow (\exists f^{N \rightarrow N})(n, f) \in b].$$

With just one application this implies Π^1_1-CA, and thus goes well beyond predicative mathematics. It is shown in section 5 how to derive the l.u.b. axiom from $Proj_1$.

NB. Part II of the notes begin with page II-4. Pages II-1 to II-3 are superseded by the preceding Introduction.

The following gives all references used in this introduction and in the notes below.

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1. Developments in VFT : set theory and the number systems.

We begin with general structural notions. The structures to be dealt with consist of a class A together with specified relations, functions and individuals in A and an "equality" relation $=_A$ on A, i.e. (in the technical algebraic sense) a congruence relation with respect to the given data. Appropriate notions of homomorphism and isomorphism are explained for such structures. Cardinal equivalence is then treated as the relation of isomorphism between structures $\langle A, =_A \rangle$. For these we have natural operations of sum, product and exponentiation, and a natural ordering relation $\langle A, =_A \rangle < \langle B, =_B \rangle$. We are then able to prove the appropriate form of Cantor's Theorem $m < 2^m$. Beyond that the subject of cardinals is studied only for countable classes, since otherwise we would have to make heavy use of the axiom of choice. We next move to the construction of the integer and rational number systems $\mathbb{Z}$ and $\mathbb{Q}$. Following that we shall take the reals $\mathbb{R}$ to be Cauchy sequences of rationals, identified in the usual way. This can be shown to satisfy usual elementary algebraic properties in VFT↾, but additional axioms will be needed to prove the completeness of $\mathbb{R}$ in §2. On the other hand, the bulk of constructive analysis can be carried out in VFT↾. To do that one starts instead with a refined notion of real number, namely as a pair consisting of a Cauchy sequence and a rate-of-convergence function. Most of §1 is taken up with basic definitions and facts having routine verifications, but the latter part deals with general considerations on the kind of additional data or refinements needed to carry out mathematics in weak theories.

1.1 Structures on classes. All single-sorted structures to be considered have the form

(1) $$\mathfrak{A} = \langle A, E, R_1, \ldots, R_m, f_1, \ldots, f_n, a_1, \ldots, a_p \rangle$$

where (for the similarity type $\langle\langle k_1, \ldots, k_m\rangle, \langle \ell_1, \ldots, \ell_n\rangle, p\rangle$) we are given classes $A, E, R_1, \ldots, R_m$, functions $f_1, \ldots, f_n$ and objects $a_1, \ldots, a_p$ satisfying:

(2) (i) $E \subseteq A^2$,

(ii) $R_i \subseteq A^{k_i}$ $(1 \leq i \leq m)$,

(iii) $f_i \in A^{\ell_i} \to A$ $(1 \leq j \leq n)$,

(iv) $a_i \in A$ $(1 \leq i \leq p)$

and where

(3) (i) E is an equivalence relation on A,

(ii) if $(x_j, y_j) \in E$ for $1 \leq j \leq k_i$ and $R_i(x_1, \ldots, x_{k_i})$ then $R_i(y_1, \ldots, y_{k_i})$, and

(iii) if $(x_j, y_j) \in E$ for $1 \leq j \leq \ell_i$ then $(f_i(x_1, \ldots, x_{\ell_i}), f_i(y_1, \ldots, y_{\ell_i})) \in E$.

In other words, E is a congruence relation for $\mathfrak{A}$.

A is called the domain of $\mathfrak{A}$ and one often loosely writes "the structure A" to refer to $\mathfrak{A}$ when given a specification (1). E is called an equality relation on A and one writes $x =_A y$ for $(x,y) \in E$. Strictly speaking this should be written $x =_{\mathfrak{A}} y$, since there are many E satisfying

(3) for any given A and additional data $R_1, \ldots, a_p$. In particular, we shall have occasion to deal with two such relations E, E' side-by-side with $E \subseteq E'$, i.e. where E is a refinement of E' (or E' is a coarsening of E). To avoid confusion in such cases we write $=_A$, $='_A$ for E, E', resp.

Often it is convenient to drop the subscript 'A' from $=_A$ altogether. This can cause confusion with the literal identity relation which we consider next, but the intended relation is usually clear from the context.

$\mathfrak{A}$ is called a discrete structure if $(x,y) \in E \Leftrightarrow x,y \in A \wedge x = y$. In this case we may write $\underset{o}{=}$ for $=_A$. By a discrete class we mean A regarded as a structure $\langle A, \underset{o}{=} \rangle$.

We have to deal occasionally with many-sorted structures given by more than one basic domain A_i, each having an equality relation $=_{A_i}$ on it. This is explained by a suitable generalization of (1)-(3), which we won't detail here.

Remark. In set theory, given a structure $\mathfrak{A}$ as specified by (1)-(3), we can form the collection A/E of equivalence classes $[x] = \{y : (x,y) \in E\}$ and then the induced structure

$$\mathfrak{A}/E = \langle A/E, R_1/E, \ldots, f_1/E, \ldots, a_1/E, \ldots \rangle$$

where for each i

$$([x_1], \ldots, [x_{k_i}]) \in (R_i/E) \Leftrightarrow (x_1, \ldots, x_{k_i}) \in R_i$$

$$(f_{i/E})([x_1], \ldots, [x_{k_i}]) = [f_i(x_1, \ldots, x_{k_i})]$$

and

$$a_i/E = [a_i] .$$

$\mathfrak{A}/E$ can be considered as a discrete structure since $[x] = [y] \Leftrightarrow (x,y) \in E$. When $E \subseteq E'$ we have the natural homomorphism of $\mathfrak{A}/E$ onto $\mathfrak{A}/E'$ by $[x]_E \mapsto [x]_{E'}$.

The basic property of equivalence classes is that they are identical or disjoint. However, this requires <u>extensionality</u> which we do not assume in this chapter, for the reasons explained in the introduction. Carrying the equality relation along as part of every structure serves the same purposes without this principle.

1.2 <u>Mappings and relations between structures</u>. Suppose given classes A, A' with equality relations $=$, $='$ resp. A function $h \in (A \rightarrow A')$ is called a <u>mapping</u> from A to A' if

(1) $$x_1 = x_2 \Rightarrow h(x_1) =' h(x_2) .$$

We write

(2) $$h: \langle A, = \rangle \rightarrow \langle A', =' \rangle$$

in this case, or more loosely $h: A \rightarrow A'$ when $=$, $='$ have been specified.

Suppose now given structures $\mathfrak{A} = \langle A, =, \ldots \rangle$ and $\mathfrak{A}' = \langle A', =', \ldots \rangle$. A mapping h from A to A' is said to be a <u>homomorphism</u> <u>from</u> $\mathfrak{A}$ <u>into</u> $\mathfrak{A}'$ if it satisfies the usual algebraic conditions for homomorphism of structures

$\langle A, E, \ldots \rangle$ into $\langle A', E', \ldots \rangle$. h is said to be a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$ (or surjection) if

(3) $$\forall y \in A' \, \exists x \in A \, (h(x) =' y).$$

h is said to be an embedding of $\mathfrak{A}$ in $\mathfrak{A}'$ (or injection) if

(4) $$h(x_1) =' h(x_2) \Rightarrow x_1 = x_2 .$$

By an isomorphism of $\mathfrak{A}$ with $\mathfrak{A}'$ we mean a pair (h, h') such that h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}'$ and h' is a homomorphism of $\mathfrak{A}'$ into $\mathfrak{A}$ with

(5) $$\forall x \in A[h'(h(x)) = x] \text{ and } \forall y \in A'[h(h'(y)) =' y].$$

In that case each of h, h' is both a surjection and an injection. We usually write (h, h^{-1}) for such pairs, though h^{-1} is not uniquely determined by h (it is only determined up to equality). When this holds we write

(6) $$(h, h^{-1}) : \mathfrak{A} \cong \mathfrak{A}'$$

or $\mathfrak{A} \cong \mathfrak{A}'$ by (h, h^{-1}).

$\mathfrak{A}$ is said to be a substructure of $\mathfrak{A}'$ if it is such in the usual algebraic sense of the word or equivalently if $A \subseteq A'$ and the identity map i_A from A to A' is an embedding of $\mathfrak{A}$ in $\mathfrak{A}'$.

1.3 Operations on classes with equality. Consider classes A, B with equality relations $=_A$, $=_B$, resp. We define an equality relation on $A \times B$ by

(1) $$(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 =_A x_2 \wedge y_1 =_B y_2 .$$

The class of all mappings from A to B is denoted both by ${}^A B$ and B^A, i.e.

(2) $${}^A B = \{f \mid f \in A \rightarrow B \wedge \forall x_1, x_2 \in A(x_1 =_A x_2 \Rightarrow f(x_1) =_B f(x_2))\}.$$

Then we define an equality relation on ${}^A B$ by

(3) $$f = g \Leftrightarrow \forall x \in A [f(x) =_B g(x)] ,$$

which is equivalent to $\forall x_1, x_2 \in A[x_1 =_A x_2 \Rightarrow f(x_1) =_B g(x_2)]$.

Now suppose that I is a discrete class and that $B = \langle B_i \rangle_{i \in I}$ is a sequence of subclasses of a class C (i.e. $B \subseteq I \times C$ and $x \in B_i \Leftrightarrow (i,x) \in B$), and $E = \langle E_i \rangle_{i \in I}$ is a sequence of subclasses of C^2 such that each E_i is an equality relation on B_i. We write $=_i$ for E_i. As explained in I.2.12, we just use $\Sigma_{i \in I} B_i$ as another notation for the sequence B. For this class $\Sigma_{i \in I} B_i$ we define the equality relation

(4) $$(i,x) = (j,y) \Leftrightarrow i = j \wedge x =_i y .$$

Next consider the class $\Pi_{i \in I} B_i$ as defined in I.2.12. The following equality relation is naturally defined for it:

(5) $$f = g \Leftrightarrow \forall i \in I(f(i) =_i g(i)).$$

Thus $I \times B$ may be treated as the special case $\Sigma_{i \in I} B$ and B^I as the special case $\Pi_{i \in I} B$.

We shall always treat $\mathbb{2}$ and $\mathbb{N}$ as discrete classes. Thus for any A, the class $\mathbb{2}^A$ consists of all functions from A into $\{0,1\}$ which preserve $=_A$; two such functions f,g are identified if $\forall x \in A(f(x) = g(x))$, i.e. if f,g are extensionally equal. Similarly for $\mathbb{N}^A$ and any I^A with I discrete.

Since $\mathbb{N}$ is discrete, countable sums and products $\Sigma_{n \in \mathbb{N}} B_n$ and $\Pi_{n \in \mathbb{N}} B_n$ fall under (4) and (5).

Now for the operations $\bigcup_{i \in I} B_i$, $\bigcap_{i \in I} B_i$ we shall only consider the simple case that each $\langle B_i, =_i \rangle$ is a substructure of $\langle C, =_C \rangle$ so we can take the equality relation on the union or intersection to be simply the restriction to that class of $=_C$.

Remark. To generalize $A \times B$ to $\Sigma_{a \in A} B_a$ and ${}^A B$ to $\Pi_{a \in A} B_a$ for non-discrete A we have to consider sequences of classes $\langle B_a, =_a \rangle$ with homomorphisms h_{ab} from B_a to B_b such that $(h_{ab}, h_{ba}) : B_a \cong B_b$ and such that the maps h_{ab} compose appropriately.

Exercise. Give this generalization in detail.

1.4 Cardinal equivalence. Two classes (with equality) A, A' are said to be cardinally equivalent if $\langle A, = \rangle \cong \langle A', =' \rangle$. We write $A \cong A'$ in this case.

Then we put $A \leq A'$ if there is an embedding of $\langle A, = \rangle$ into $\langle A', =' \rangle$, and $A < A'$ if $A \leq A'$ but $A \not\equiv A'$. A is said to be <u>countable</u> if A is empty or there exists a surjection of $\mathbb{N}$ on A (i.e. on $\langle A, = \rangle$). A structure is called countable if its domain is countable.

Using an isomorphism of $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$ it is easily proved that if A, B are countable then so is $A \times B$. Thus we obtain for each (particular) n a surjection h_n of $\mathbb{N}$ on A^n. Next if I is discrete and countable and $\langle h_i \rangle_{i \in I}$ is a given sequence of surjections h_i of $\mathbb{N}$ on B_i then $\Sigma_{i \in I} B_i$ is countable. Finally, the same holds for $\bigcup_{i \in I} B_i$ (where applicable), since $\Sigma_{i \in I} B_i$ is mapped onto $\bigcup_{i \in I} B_i$ by $h(i,x) = x$.

The class $2^{\mathbb{N}}$ is <u>uncountable</u>; the proof is by contradiction, by means of the usual Cantor diagonal argument. It follows that $\mathbb{N} < 2^{\mathbb{N}}$; similarly $\mathbb{N} < \mathbb{N}^{\mathbb{N}}$. More generally $A < 2^A$ for any A.

<u>Remark</u>. VFT is too weak to prove that every countable class is equivalent to an initial segment of $\mathbb{N}$ (why?); this will be proved in $\mathrm{VFT}{\restriction} + (\mu)$. Also we must wait to prove there that every sub<u>set</u> of a countable class is countable (though not necessarily every sub<u>class</u>).

<u>Exercises</u> (In $\mathrm{VFT}{\restriction}$).

(i) Prove $A < 2^A$.

(ii) The class $A^{<\omega}$ of finite sequences from A was defined in I.5.8 as $\{(n,g) \mid g \in \mathbb{N} \to A \wedge \forall m \geq n \; g(m) = g(n)\}$. Show that if A is countable so also is $A^{<\omega}$.

1.5 Ordered structures and algebraic structures. A structure $\langle A, =, \leq \rangle$ is said to be partially ordered if (i) $x = y \Leftrightarrow x \leq y \wedge y \leq x$, and (ii) $\leq$ is transitive. It is said to be linearly ordered if further (iii) $x \leq y \vee y \leq x$ for each $x, y \in A$. We write $x < y$ for $x \leq y \wedge x \neq y$.

By a group we mean a structure $\langle G, =, \circ, ^{-1}, e\rangle$ such that (i) $\forall x,y,z \in G[(x \circ y) \circ z = x \circ (y \circ z)]$, (ii) $\forall x \in G\,(x \circ e = e \circ x = x)$, (iii) $\forall x \in G[(x \circ x^{-1}) = (x^{-1} \circ x) = e]$. We define similarly the notions of ring, integral domain, field, ordered domain and ordered field. In the last it is assumed we have an operation $(\ )^{-1}$ such that for each $x \neq 0$, $x \circ x^{-1} = 1$.

By a vector space we mean a two-sorted structure consisting of a group $\langle X, =_X, \oplus, \bar{0}\rangle$ (the vectors) and a field $\langle K, =_K, +, -, \circ, ^{-1}, 0, 1\rangle$ together with a binary mapping $m: K \times X \to X$ (scalar multiplication), satisfying the usual vector space laws. X is then said to form a vector space over K.

1.6 The integers $\mathbb{Z}$. We take $\mathbb{Z}$ to be $\mathbb{N} \times \mathbb{N}$ with the equality relation

(1) $$(n_1, m_1) =_{\mathbb{Z}} (n_2, m_2) \Leftrightarrow n_1 + m_2 = m_2 + n_1 .$$

Then $\mathbb{Z}$ forms an ordered integral domain using the following structure:

(2) (i) $(n_1, m_1) + (n_2, m_2) = (n_1 + n_2, m_1 + m_2)$

(ii) $-(n_1, m_1) = (m_1, n_1)$

(iii) $(n_1, m_1) \circ (n_2, m_2) = (n_1 n_2 + m_1 m_2, n_1 m_2 + m_1 n_2)$

(iv) $0_{\mathbb{Z}} = (0,0)$

(v) $1_{\mathbb{Z}} = (1,0)$

(vi) $(n_1,m_1) < (n_2,m_2) \Leftrightarrow n_1 + m_2 < m_1 + n_2$.

Further we have an embedding of $\mathbb{N}$ in $\mathbb{Z}$ by the map $n \mapsto n_{\mathbb{Z}} = (n,0)$. $\mathbb{Z}$ is generated by the image of $\mathbb{N}$ under this injection, since $(n,m) =_{\mathbb{Z}} (n,0) + (-(m,0))$. We shall identify n with $n_{\mathbb{Z}}$ and $\mathbb{N}$ with its image under this map, i.e. treat $\mathbb{N}$ as a subclass of $\mathbb{Z}$. Thus for each $x \in \mathbb{Z}$ either $x \in \mathbb{N}$ or $(-x) \in \mathbb{N}$. Furthermore, $\mathbb{N}$ forms a substructure of $\mathbb{Z}$ under (2)(i),(iii)-(vi). Note that $x = (m,n) = (m-n)$ is in the image of $\mathbb{N}$ just in case $m \geq n$. We now drop the subscript '$\mathbb{Z}$' from $=_{\mathbb{Z}}$.

1.7 <u>The rationals</u> $\mathbb{Q}$. We take $\mathbb{Q}$ to be $\mathbb{Z} \times (\mathbb{N} - \{0\})$ with the equality relation

(1) $$(x_1, n_1) =_{\mathbb{Q}} (x_2, n_2) \Leftrightarrow x_1 n_2 = x_2 n_1 .$$

Then $\mathbb{Q}$ <u>forms an ordered field</u> using the following structure:

(2) (i) $(x_1,n_1) + (x_2,n_2) = (x_1 n_2 + n_1 x_2 , n_1 n_2)$

(ii) $-(x,n) = (-x, n)$

(iii) $(x_1,n_1) \cdot (x_2,n_2) = (x_1 x_2 , n_1 n_2)$

(iv) $$(x,n)^{-1} = \begin{cases} (n,x) & \text{if } x > 0 \\ (0,1) & \text{if } x = 0 \\ (-n,-x) & \text{if } x < 0 . \end{cases}$$

(v) $(x_1,n_1) < (x_2,n_2) \Leftrightarrow x_1 n_2 < x_2 n_1$.

(Note that $(0,n)^{-1}$ is just given a conventional value; for $x \neq 0$, $(x,n)\cdot(x,n)^{-1} =_{\mathbb{Q}} (1,1)$.) Now we have an injection of $\mathbb{Z}$ in $\mathbb{Q}$ by $x \mapsto x_{\mathbb{Q}} = (x,1)$. We identify x with $x_{\mathbb{Q}}$ and $\mathbb{Z}$ with its image under this map, i.e. treat $\mathbb{Z}$ as a subclass of $\mathbb{Q}$. $\mathbb{Z}$ then forms a substructure of $\mathbb{Q}$ with respect to (2)(i) - (iii) and (v). (x,n) is in $\mathbb{Z}$ just in case $n|x$. $\mathbb{Q}$ is generated by $\mathbb{Z}$ with $(x,n) =_{\mathbb{Q}} (x,1)\cdot(n,1)^{-1}$. The subscript '$\mathbb{Q}$' is now dropped from $=_{\mathbb{Q}}$. We usually use $r, s, \ldots$ to range over $\mathbb{Q}$. For $r, s \in \mathbb{Q}$ with $s \neq 0$, $r/s = \frac{r}{s} = rs^{-1}$. The absolute value $|r|$ is defined as usual.

1.8 <u>The reals</u> $\mathbb{R}$ <u>as Cauchy sequences of rationals</u>. We use $\langle r_n \rangle_{n \in \mathbb{N}}$ or simply $\langle r_n \rangle$ to denote a sequence of rationals. The class of all such sequences is $\mathbb{Q}^{\mathbb{N}}$. $\mathbb{R}$ is defined to be the sub-class of $\mathbb{Q}^{\mathbb{N}}$ consisting of all $\langle r_n \rangle$ such that

$$(1) \qquad \forall m > 0 \ \exists n \ \forall k_1, k_2 \geq n \ (|r_{k_1} - r_{k_2}| < \tfrac{1}{m}) \ .$$

The equality relation assigned to $\mathbb{R}$ is

$$(2) \qquad \langle r_n \rangle =_{\mathbb{R}} \langle s_n \rangle \Leftrightarrow \forall m > 0 \ \exists n \ \forall k \geq n \ (|r_k - s_k| < \tfrac{1}{m}) \ ,$$

which is easily verified to be an equivalence relation on $\mathbb{R}$. The algebraic operations on $\mathbb{R}$ and its ordering are then defined by:

(3) (i) $\langle r_n\rangle + \langle s_n\rangle = \langle r_n + s_n\rangle$

(ii) $-\langle r_n\rangle = \langle -r_n\rangle$

(iii) $\langle r_n\rangle \cdot \langle s_n\rangle = \langle r_n \cdot s_n\rangle$

(iv) $\langle r_n\rangle^{-1} = \langle r_n^{-1}\rangle$

(v) $\langle r_n\rangle < \langle s_n\rangle \Leftrightarrow \exists m > 0\, \exists n\, \forall k \geq n\, \{(s_k - r_k) > \frac{1}{m}\}$.

We have an injection of $\mathbb{Q}$ in $\mathbb{R}$ by $r \mapsto r_{\mathbb{R}} = \langle r\rangle_{n \in \mathbb{N}}$, i.e. the sequence with constant value r. We shall identify r with $r_{\mathbb{R}}$ and $\mathbb{Q}$ with its image under this map. $\langle \mathbb{R}, =_{\mathbb{R}}, +, -, \cdot, 0, 1\rangle$ is easily verified to form a commutative ring with unity. To show that it forms a field with the inverse operation, one argues as follows. Suppose $\langle r_n\rangle \neq 0$; it is to be shown that $\langle r_n\rangle^{-1} \in \mathbb{R}$ and $\langle r_n\rangle \cdot \langle r_n\rangle^{-1} = 1$. First we claim that $0 < \langle r_n\rangle$ or $\langle r_n\rangle < 0$. For if not then for each $m > 0, n$ we have $\exists k_1 \geq n\{r_{k_1} \leq \frac{1}{m}\}$ and $\exists k_2 \geq n\{-r_{k_2} \leq \frac{1}{m}\}$. Applying the Cauchy property (1) it follows that for each m there exists p with $\forall k \geq p\{-\frac{2}{m} < r_k < \frac{2}{m}\}$ and hence $\langle r_n\rangle = 0$. To show that $\langle r_n\rangle^{-1} \in \mathbb{R}$ when $0 < \langle r_n\rangle$, use

$$|r_k^{-1} - r_\ell^{-1}| = \left|\frac{1}{r_k} - \frac{1}{r_\ell}\right| = \left|\frac{r_\ell - r_k}{r_k r_\ell}\right| \leq M^2 \cdot |r_\ell - r_k| \quad \text{when } \frac{1}{M} \leq r_k, r_\ell\,;$$

similarly $\langle r_n\rangle^{-1} \in \mathbb{R}$ follows from $\langle r_n\rangle < 0$. It is easily checked that $\langle r_n\rangle \cdot \langle r_n\rangle^{-1} = 1$ in either case. Using this line of reasoning we are able to establish that $\mathbb{R}$ <u>forms an ordered field</u> under the structure (2), (3), and that $\mathbb{Q}$ is a subfield of $\mathbb{R}$ under the identification described above. Furthermore, $\mathbb{R}$ may be verified to have the Archimedean property

(4) $$\forall x \in \mathbb{R}[x > 0 \Rightarrow \exists m > 0(\frac{1}{m} < x)]$$

which is equivalent to the density of $\mathbb{Q}$ in $\mathbb{R}$:

(5) $$\forall x,y \in \mathbb{R}\{x < y \Rightarrow \exists r \in \mathbb{Q}(x < r < y)\} .$$

Remark. A Cauchy sequence of reals is an element $\langle x_n \rangle$ of $\mathbb{R}^{\mathbb{N}}$ satisfying $\forall m > 0\ \exists n\ \forall k_1, k_2 \geq n\{|x_{k_1} - x_{k_2}| < \frac{1}{m}\}$. Here each x_n is itself a sequence $\langle r_{n,k} \rangle_{k \in \mathbb{N}}$. To prove the completeness of $\mathbb{R}$ we need a function $f(n,m)$ which associates with each $m > 0$ and n a number $p = f(n,m)$ satisfying $|r_{n,k_1} - r_{n,k_2}| < \frac{1}{m}$ for all $k_1, k_2 \geq p$. The existence of such f cannot be proved in VFT$\upharpoonright$, but is easily established in VFT$\upharpoonright$ + (μ); this will be subsumed in § 2 under the proof of local sequential compactness of $\mathbb{R}$.

Exercise. Carry out in detail the proof in VFT$\upharpoonright$ that $\langle \mathbb{R}, =_{\mathbb{R}}, +, -, \cdot, ^{-1}, <, 0, 1 \rangle$ is an Archimedean ordered field containing $\mathbb{Q}$ as substructure.

1.9 The constructive treatment of $\mathbb{R}$. There are several approaches to constructive mathematics, most notably those of Brouwer, Markov and Bishop; cf. Troelstra 1977. The one closest to ordinary mathematical practice is that of Bishop 1967,* and it is that to which we shall refer in contrast to the non-constructive work done here. Constructive mathematics requires all existential statements to be justified by explicit effective constructions

* Revised and extended in Bishop and Bridges 1985.

and rejects the use of the law of excluded middle. Almost all of Bishop 1967 can be carried out in VFT↾ using only intuitionistic logic. The verification of this would take us too far afield.

The proof just sketched in 1.8 that if $x \in \mathbb{R}$ and $x \neq_{\mathbb{R}} 0$ then $0 < x$ or $x < 0$ proceeds by contradiction and hence requires classical logic. Thus in constructive mathematics, one uses another definition of $\mathbb{R}$, which we shall label here $\tilde{\mathbb{R}}$. The elements of $\tilde{\mathbb{R}}$ are the pairs $(\langle r_n \rangle, c)$ where $c : \mathbb{N} \to \mathbb{N}$ and

$$(1) \qquad \forall m > 0 \quad \forall k_1, k_2 \geq c(m)\{|r_{k_1} - r_{k_2}| < \frac{1}{m}\}.$$

c is called a <u>rate</u> (or <u>modulus</u>) <u>of</u> <u>convergence</u> <u>function</u> <u>for</u> $\langle r_n \rangle$. The same $\langle r_n \rangle$ will of course have an infinity of such c if it has any at all. Two members of $\tilde{\mathbb{R}}$ are identified by

$$(2) \qquad (\langle r_n \rangle, c_1) =_{\tilde{\mathbb{R}}} (\langle s_n \rangle, c_2) \Leftrightarrow \langle r_n \rangle =_{\mathbb{R}} \langle s_n \rangle.$$

It follows from (2) that there is a function c such that $\forall m > 0\ \forall k \geq c(m)\{|r_k - s_k| < \frac{1}{m}\}$. Actually the definition of reals in Bishop 1967 is a special case of (1); he considers only sequences $\langle r_n \rangle$ for which

$$(1)' \qquad \forall k_1, k_2 > 0\ \{|r_{k_1} - r_{k_2}| < \frac{1}{k_1} + \frac{1}{k_2}\};$$

for these we can take $c(m) = 2m$ to satisfy (1). Conversely any $\langle r_n \rangle$ satisfying (1) with suitable c contains a subsequence $\langle r'_n \rangle$ satisfying

(1)' such that $\langle r_n \rangle =_{\mathbb{R}} \langle r'_n \rangle$. With either approach to $\tilde{\mathbb{R}}$ we can define suitable operations $+, -, \cdot$ and prove that $\langle \tilde{\mathbb{R}}, =_{\tilde{\mathbb{R}}}, +, -, \cdot, 0, 1 \rangle$ is a commutative ring. The property of inverse is established constructively only for those reals x which are separated from 0, i.e. for which (by definition) $0 < x$ or $x < 0$. (Without the law of excluded middle this is not equivalent to $x \neq 0$.)

The algebra of $\tilde{\mathbb{R}}$ can be given in (intuitionistic) VFT↾ following Bishop 1967 pp.15-25. But now also the completeness of $\tilde{\mathbb{R}}$ can be proved in this theory (loc. cit. p.27). With reference to the definition (1), the reason is that now a sequence $\langle x_n \rangle$ of reals is given by a sequence of pairs $(\langle r_{n,k} \rangle_{k \in \mathbb{N}}, c_n)$ where each c_n is a rate of convergence function for x_n. Using this one may obtain constructively the existence of a limit x for the sequence $\langle x_n \rangle$.

One additional feature of the constructive development of analysis should be mentioned before we pass on to the next section. Suppose given $a, b \in \tilde{\mathbb{R}}$ with $a < b$; let $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$. A function $f : [a,b] \to \tilde{\mathbb{R}}$ is said to be uniformly continuous if there is a function $w : \mathbb{Q}_+ \to \mathbb{Q}_+$ such that

$$(3) \qquad \forall \epsilon > 0 \; \forall x,y \in [a,b] \{ |x-y| < w(\epsilon) \Rightarrow |f(x)-f(y)| < \epsilon \} .$$

w is called a modulus of (uniform) continuity function for f. When carrying out operations on continuous functions such as integrating $\int_a^b f(x)dx$ we need to use the information w as well as f; thus strictly speaking $\int_a^b f(x)dx$ is a function $I(f,w)$ (for given a,b). For this reason, the class $\tilde{C}([a,b], \tilde{\mathbb{R}})$ of uniformly continuous function from $[a,b]$ to $\tilde{\mathbb{R}}$ is defined by Bishop to consist of all pairs (f,w) satisfying (3)

1.10 Explicit presentation of mathematical objects. By a presentation of a class A we mean a pair (A',h) where $h: A' \to A$ is a surjection; when $a = h(a')$ we say that a is labeled (or presented) by a'. The terminology comes from algebra (e.g. finitely presented groups) and topology (e.g. surfaces presented by complexes). A familiar and typical example is a manifold A with atlas (A', h) which supplies (possibly) overlapping co-ordinate systems which cover A. In such algebraic or geometric examples of presentations we may think of a labeling $h(a') = a$ as an explicit description of how a is generated or located. More generally, in explicit reformulations of classical mathematics we must often work with explicit information a' which tells us how $a = h(a')$ "comes to be" an element of A. For simplicity we can always take this labeling in the form $a' = (a,w)$ where w is the witnessing or side-information which verifies (in a suitable sense) that $a \in A$, so that h is simply the first projection function p_1. We have seen two examples of this in connection with constructive mathematics in the previous section, namely the presentation of $\mathbb{R}$ by $\tilde{\mathbb{R}}$ and of $C([a,b], \mathbb{R})$ by $\tilde{C}([a,b], \tilde{\mathbb{R}})$. In both cases the definition of A calls for an existential requirement to be met, which is provided by the witnessing information. As we shall see, for the mathematics carried out in $VFT\upharpoonright + (\mu)$, this type of presentation is not needed in the initial part of classical analysis, but will be needed when we pass to topology (where an open set will be presented as a countable union of basic open sets) and measure theory (where a measurable set will be presented with its complement as a limit (up to measure 0) of open sets).

1.11 Classes with countably presented members. When we have a surjection $h : \mathbb{N}^{\mathbb{N}} \to A$, the elements of A are said to be countably presented, since each member of A can be described by a countable amount of information, namely an $f \in \mathbb{N}^{\mathbb{N}}$ such that $h(f) = a$. We shall meet many examples of such, e.g.: (i) the class $\mathbb{R}$ of real numbers, as already defined in 1.9, (ii) the class of continuous functions on $\mathbb{R}$ to $\mathbb{R}$ (each of which is determined by its values at the rationals), (iii) the class of sequences $\langle a_n \rangle$ of real numbers, (iv) the class of analytic functions (given by power series $\Sigma_{n=1}^{\infty} a_n x^n$), (v) the class of Lebesgue measurable functions, (vi) the class of continuous linear operators on a Hilbert space of countable dimension, etc.

The ubiquity of such classes (and their associated structures) explains informally why so much of analysis can be reduced to 2nd order terms, even though the objects considered are of prima-facie higher order (the functions between classes of type n forming a class of type $n+1$). However, attempts to formalize analysis in 2nd order terms have usually been rather forced. Instead we formalize the mathematics directly in finite-type theories such as VFT↾ and its extensions and then use general logical methods to reduce these to second-order theories.

2. <u>Developments in</u> VFT $+ (\mu)$: <u>set theory and topological spaces</u>. The axiom (μ) which provides for unbounded minimalization is now adjoined; this allows us to decide statements built up by numerical quantification. The use of the new axiom in set theory is distinctively illustrated by proofs of König's tree theorem and the Cantor-Schröder-Bernstein theorem.

The main work of this section has to do with topological spaces with countable basis (and, among these, principally Hausdorff and separable spaces). These are closed under formation of subspaces, products and countable powers so that starting with the spaces $\mathbf{2}, \mathbb{N}$ and $\mathbb{R}$ we obtain in particular Cantor space $\mathbf{2}^{\mathbb{N}}$, Baire space $\mathbb{N}^{\mathbb{N}}$ and the Euclidean spaces $\mathbb{R}^n$. Topologically the complex numbers $\mathbb{C}$ appear as $\mathbb{R}^2$. Compactness is first studied in sequential form; it is shown to be preserved under products and powers. (Compactness in terms of open coverings requires more careful considerations and is not taken up until later.)

We next move to the topology of $\mathbb{R}$ and from there to metric spaces generally. The completeness of $\mathbb{R}^n$ follows from the sequential compactness of closed bounded subspaces. In particular for $\mathbb{R}$ we obtain the existence of least upper bounds (and greatest lower bounds) for bounded <u>sequences</u>. On the other hand the least upper bound property for <u>sets</u> cannot be derived in VFT $+ (\mu)$ and needs the essential new axiom (Proj_1), which will be considered in §5.

Two results of interest for complete metric spaces are the Baire category theorem and the contraction mapping theorem. Both are established here under the additional hypothesis of separability, which is met in all the spaces of concern to us. From a logical point of view it is of interest that the standard proof of the contraction mapping theorem can be given directly in VFT but must be modified carefully in order to obtain it in $\mathrm{VFT}\upharpoonright + (\mu)$.

At the end of the section we return to the subject of compactness as defined in terms of open coverings. It is shown that sequential compactness is equivalent to a strong form of compactness for countable open covers when a certain additional property ("cover-witness") of the space is met. This property holds for 2 and closed intervals in $\mathbb{R}$ and is closed under products and countable powers. On the other hand, we do not obtain compactness for arbitrary open covers without assuming (Proj_1).

2.1 Subsets of countable classes. Suppose A with the equality relation $=$ is countable and non-empty, i.e. that there is a surjection f in $A^{\mathbb{N}}$. Then any $a \in \mathcal{S}(A)$ is also countable. For, if $\underline{a}$ is non-empty, we can define a sequence $\langle k_n \rangle$ of natural numbers by recursion $(\underset{\sim}{r}_{\mathbb{N}})$ as follows, in such a way that the function $g(n) = f(k_n)$ enumerates $\underline{a}$:

(1) (i) $k_o = 0$

$$\text{(ii)}\quad k_{n+1} = \begin{cases} \mu m(m > k_n \wedge f(m) \in a) & \text{if } \exists m > k_n(f(m) \in a) \\ k_n & \text{otherwise.} \end{cases}$$

On the other hand it cannot be proved that any subclass A_1 of A is countable.

2.2 The König tree theorem. This concerns finitely branching trees; each such is isomorphic to one labelled in $\mathbb{N}$, so we consider only subtrees of $\mathbb{N}$. Here we use the notions and notation of I.5.7. In addition, let $b \restriction s$ represent the tree below s in b, i.e. the set of t with $s * t \in b$. Given any s it is decidable (using (μ) whether $b \restriction s$ is infinite or not, i.e. whether or not $\forall n \geq lh(s) \exists t \in b(s \subset t \wedge lh(t) = n)$.

We suppose b is an infinite finitely branching tree in $\mathbb{N}$, so that for each $s \in b$ there are at most finitely many m with $s * \langle m \rangle \in b$. We shall define a function f representing an infinite branch through b by recursion $(\underset{\sim}{r}_{\mathbb{N}})$ as follows:

(1) (i) $f(0) = \mu m (b \restriction \langle m \rangle$ is infinite)

(ii) $f(n+1) = \mu m(b \restriction (\bar{f}(n) * \langle m \rangle)$ is infinite).

It is proved by induction on n that

(2) $$b \restriction \bar{f}(n) \text{ is infinite.}$$

This property defines a set so we are using only the principle $I_{\mathbb{N}}$ of set-induction. To carry out the induction step, if $b \restriction \bar{f}(n)$ is infinite there is at least one m such that $b \restriction (\bar{f}(n) * \langle m \rangle)$ infinite since by assumption $\exists k \forall m[\bar{f}(n) * \langle m \rangle \in b \Rightarrow m \leq k]$. In particular from (2) we have

(3) $$\forall n[\bar{f}(n) \in b],$$

i.e. f is an infinite branch through b. It should be noted that the construction (1) is <u>uniform in</u> b, i.e. that f is associated with b by a function $f = F(b)$.

2.3 <u>The Cantor-Schröder-Bernstein theorem</u>. Given two non-empty classes (with equality) A, B, their direct sum $A \dot{+} B$ may be defined as a subclass of $A \times B \times 2$; for this purpose, fix any $a_o \in A$, $b_o \in B$, and take

(1) $$A \dot{+} B = \{(a,b,i) \in A \times B \times 2 \mid a = a_o \wedge i = 1 \vee b = b_o \wedge i = 0\}$$

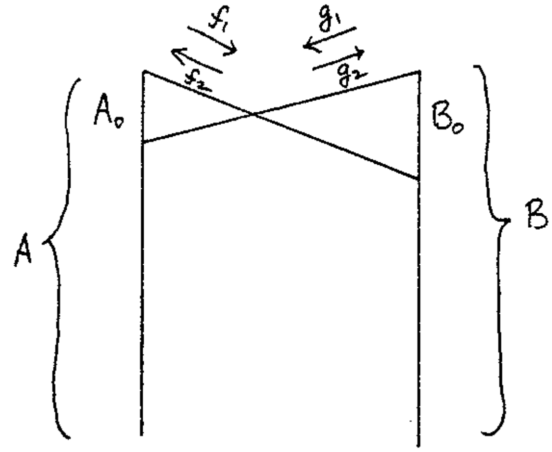
then we put

(2) $$A \leq^* B \Leftrightarrow \text{ for some } C, A \dot{+} C \cong B .$$

The form of the Cantor-Schröder-Bernstein theorem which we shall prove here is:

(3) $$A \leq^* B \wedge B \leq^* A \Rightarrow A \cong B.$$

(The form $A \leq B \wedge B \leq A \Rightarrow A \cong B$ cannot be proved in VFT + (μ) as will be shown later.) It is seen that under the hypotheses $A \leq^* B$, $B \leq^* A$, there exist $A_o \subseteq A$, $B_o \subseteq B$, functions $a_o \in 2^A$, $b_o \in 2^B$, and functions $f_1, g_2 \in B^A$ and $f_2, g_1 \in A^B$ such that



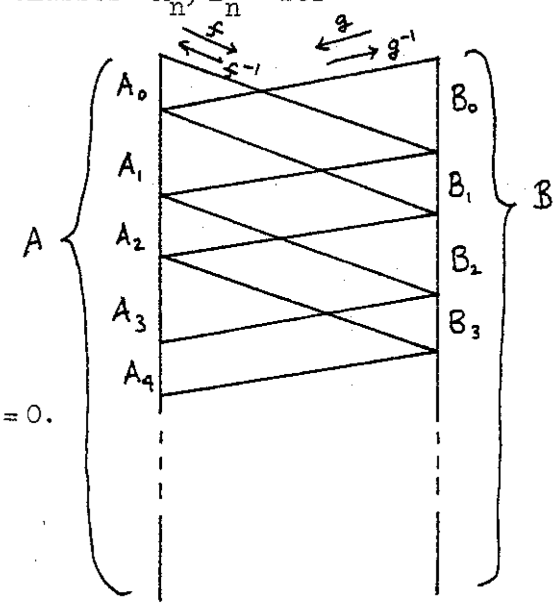
(4) (i) for $x \in A$, $a_o(x) = 0 \Leftrightarrow x \in A_o$

(ii) for $y \in B$, $b_o(y) = 0 \Leftrightarrow y \in B_o$

(iii) f_1, f_2 determine an isomorphism of A with $B\text{-}B_o$

(iv) g_1, g_2 determine an isomorphism of B with $A\text{-}A_o$.

We write f for f_1 and f^{-1} for f_2, g for g_1 and g^{-1} for g_2. (4)(i) says that (on A) membership in A_o is decided; similarly for b, B_o. Now we shall define functions a, b and classes A_n, B_n for $n \geq 1$ to fill in the picture at the right:



(5) (i) $a(0,x) = a_o(x) \quad b(0,y) = b_o(y)$

(ii) $a(n+1,x) = (1-a_o(x)) + b(n,g^{-1}(x))$

$b(n+1,y) = (1-b_o(y)) + a(n,f^{-1}(y))$

(iii) $x \in A_n \Leftrightarrow a(n,x) = 0, \quad y \in B_n \Leftrightarrow b(n,y) = 0.$

Thus membership in the A_n's and B_n's is uniformly decided.

The functions a^*, b^* take values in $\mathbb{N}$ and are produced simultaneously by $\underset{\sim}{r}_{\mathbb{N}}$. It is seen that for $x \in A$, $y \in B$

(6) (i) $x \in A_{n+1} \Leftrightarrow x \notin A_o \wedge g^{-1}(x) \in B_n$ and

(ii) $y \in B_{n+1} \Leftrightarrow y \notin B_o \wedge f^{-1}(y) \in A_n$.

The A_n's do not necessarily exhaust A; similarly for the B_n's and B. We take

(7) $$A_\omega = \bigcup_{n \in \mathbb{N}} A_n, \quad B_\omega = \bigcup_{n \in \mathbb{N}} B_\omega, \quad A' = A - A_\omega, \quad B' = B - B_\omega .$$

Using (μ) we have that membership in these classes is decided by:

(8) $$x \in A_\omega \Leftrightarrow \exists n[a(n,x) = 0], \quad y \in B_\omega \Leftrightarrow \exists n[b(n,y) = 0] .$$

To complete the proof, the desired isomorphism h with inverse h^{-1} will be defined as follows, so as to match $A_o \leftrightarrow B_1$, $A_1 \leftrightarrow B_o$, $A_2 \leftrightarrow B_3$, $A_3 \leftrightarrow B_2, \ldots,$ and $A' \leftrightarrow B'$:

(9) (i) $$h(x) = \begin{cases} f(x) & \text{if } x \in A' \text{ or } x \in A_{2n} \text{ for some } n \\ g^{-1}(x) & \text{if } x \in A_{2n+1} \text{ for some } n . \end{cases}$$

(ii) $$h^{-1}(y) = \begin{cases} g(y) & \text{if } y \in B' \text{ or } y \in B_{2n} \text{ for some } n \\ f^{-1}(y) & \text{if } y \in B_{2n+1} \text{ for some } n . \end{cases}$$

It may be seen that h, h^{-1} make $A \cong B$.

This result has been included to illustrate the kind of work which may be done in set theory beyond 1.4 when we adjoin (μ). It is not needed below.

2.4 <u>Topological spaces with countable basis</u>. The structures used here are of the form $\mathfrak{A} = \langle A, =, \langle B_n \rangle_{n \in \mathbb{N}} \rangle$ where $=$ is an equality relation on A and each $B_n \subseteq A$. The sequence $B = \langle B_n \rangle$ is considered to be given as a subclass of $\mathbb{N} \times A$, with $B_n = \{x \in A \mid (n,x) \in B\}$. In this sense we are dealing with a two-sorted structure over A and $\mathbb{N}$. The structure on A is called a <u>(topological) space</u> if it satisfies the following conditions:

(1) (i) B is a set, i.e. the relation $x \in B_n$ is decidable,

(ii) $A \subseteq \bigcup_n B_n$, and

(iii) for each n_0, n_1 and $x \in B_{n_0} \cap B_{n_1}$ we can find n with

$$x \in B_n \subseteq B_{n_0} \cap B_{n_1} .$$

The B_n's form a <u>basis</u> for a topology on A and are called the <u>basic open sets</u> in A. Thus only spaces with countable basis are considered here. An <u>open set</u> is defined to be a union of basic open sets. By a <u>presentation of an open set</u> we mean any subset g of $\mathbb{N}$; the open set presented by g is

(2) $$G = \bigcup_{n \in g} B_n ,$$

which is denoted $U(g)$. Thus the open sets are the $U(g)$'s for $g \in \mathcal{S}(\mathbb{N})$.

By the basic closed sets of A we mean the sets $C_n = A - B_n$. By a presentation of a closed set we mean any subset f of $\mathbb{N}$; the closed set F represented by f is given by

(3) $$F = \bigcap_{n \in f} C_n ,$$

which is also denoted $U'(f)$. The closed sets of A are the $U'(f)$'s for $f \in \mathcal{S}(\mathbb{N})$. Thus any subset a of $\mathbb{N}$ serves to represent both an open and a closed set by $U(a)$ and $U'(a)$, resp.; the relation is obviously $U'(a) \equiv A - U(a)$.

The open sets are closed under countable unions in the following sense: given a sequence $\langle g_n \rangle$ of presentations of open sets $G_n = U(g_n)$, we obtain a presentation g of $\bigcup_n G_n$ simply by $g = \bigcup_n g_n$; for, $U(g) \equiv \bigcup_{n \in g} B_n \equiv \bigcup_n \bigcup_{m \in g} B_m \equiv \bigcup_n G_n$. Similarly the closed sets are closed under countable intersections. It is easily seen from (1)(iii) that the open (closed) sets are closed under finite intersections (unions).

Remark. In set-theoretical topology we also have closure of the open sets under arbitrary unions. Formulated in terms of presentations, this is the assertion that if $a \subseteq \mathcal{S}(\mathbb{N})$ then $G = \bigcup_{g \in a} U(g)$ is open. To prove this one simply takes $g_1 = \bigcup_{g \varepsilon a} g$ so $G \equiv U(g_1)$. But this cannot be carried out in VFT + (μ), since in effect $n \varepsilon g_1 \Leftrightarrow \exists g \varepsilon \mathbb{S}(\mathbb{N})[g \varepsilon a \wedge n \varepsilon g]$ requires quantification over $\mathbb{S}(\mathbb{N})$. The statement does follow simply from the additional axiom Proj_1 as will be discussed in § 5.

Exercise. Prove that the open sets are closed under finite intersections.

2.5 Hausdorff spaces; limits. The space A is said to be Hausdorff if it satisfies

(1) for each x,y with $x \neq y$ there exists n,m with $x \varepsilon B_n$, $y \varepsilon B_m$ and $B_n \cap B_m$ empty.

From now on all spaces are assumed to be Hausdorff. Note that by our general requirements on structures with equality, the relation = is supposed to be a congruence relation for $B = \langle B_n \rangle$, i.e.

(2) $$x = y \Rightarrow (x \varepsilon B_n \Rightarrow y \varepsilon B_n).$$

Conversely in a Hausdorff space we have

(3) $$\forall n(x \varepsilon B_n \Rightarrow y \varepsilon B_n) \Rightarrow x = y ,$$

which is a kind of extensionality principle. Given $x_1,\ldots,x_m \varepsilon A$, let $\{x_1,\ldots,x_m\} = \{y \varepsilon A | y=x_1 \vee \ldots \vee y = x_m\}$. Each $\{x\}$ is closed since $\{x\} \equiv U'(a)$ for $a = \{n | x \varepsilon C_n\}$. Hence each $\{x_1,\ldots,x_m\}$ is also closed.

A is said to be a discrete space if each $\{x\}$ is open. $\mathbb{N}$ is always considered as a discrete space, and any countable A with decidable equality may be treated as a discrete space.

Given any space A and sequence $\langle x_m \rangle$ from A we put

$$(4) \qquad \lim_n x_n = x \Leftrightarrow \forall m[x \in B_m \Rightarrow \exists n \, \forall k \geq n(x_n \in B_m)].$$

$\langle x_n \rangle$ is called convergent if $\exists x(\lim_n x_n = x)$. Note that the relation (4) is decidable. The Hausdorff property insures uniqueness of limits:

$$(5) \qquad \lim_n x_n = x \wedge \lim_n x_n = y \Rightarrow x=y .$$

Suppose $X \subseteq A$; we say that x is a limit point of X if

$$(6) \qquad \forall n[x \in B_n \Rightarrow \exists y(y \in X \cap B_n)] ,$$

and that x is a strong limit point of X if we have a function f such that

$$(7) \qquad \forall n[x \in B_n \Rightarrow f(n) \in X \cap B_n].$$

Let $\bar{X}(\bar{X}^{(s)})$ be the class of (strong) limit points of X. Obviously $X \subseteq \bar{X}^{(s)} \subseteq \bar{X}$, but we cannot prove $\bar{X} \subseteq \bar{X}^{(s)}$ in general without the axiom of choice. When X is countable AC is dispensable and we have $\bar{X} \equiv \bar{X}^{(s)}$ in that case. It may be seen that for any X,

(8) $$x \in \bar{X}^{(s)} \Leftrightarrow \exists \langle x_n \rangle \{\forall n(x_n \in X) \wedge \lim_n x_n = x\} .$$

X is said to be closed under limits if $\bar{X}^{(s)} \subseteq X$. Each closed set $F = U'(f)$ is closed under limits. For suppose $\lim_n x_n = x$ where each $x_n \in F = \bigcap_{m \in f} (A - B_m)$. If $x \notin F$ then for some $m \in f$ we have $x \in B_m$; but then some $x_n \in B_m$, which is a contradiction. However, it cannot be proved in $VFT + (\mu)$ that each X which is closed under limits is closed in the sense of 2.4, though that is set-theoretically true. (Where does the usual proof break down in $VFT + (\mu)$?)

X is said to be dense (strongly dense) in A if $A \subseteq \bar{X}$ ($A \subseteq \bar{X}^{(s)}$). A is said to be a separable space if it has a countable dense subset $X = \{q_n\}_{n \in \mathbb{N}}$; then X is also strongly dense in A.

Exercise Prove (8).

2.6 Subspaces, products and powers. Suppose given a space A with basic open sets $\langle B_n \rangle$, and that $A_o \subseteq A$. The topology of A may be relativized to A_o by taking the collection $\langle B_n \cap A_o \rangle$ as the basic open sets of A_o. In this sense any subclass of A determines a subspace.

Given spaces A, A' with basic open sets $\langle B_n \rangle, \langle B'_n \rangle$, resp., we form a structure on $A \times A'$ as follows. The equality relation is $=_{A \times A'}$, and the basic open sets are given by the sequence $\langle B_n \times B'_m \rangle_{(n,m) \in \mathbb{N}^2}$ of basic "rectangles" which is reduced to a countable sequence by an enumeration of $\mathbb{N}^2$. Obviously the relation $(x,y) \in B_n \times B'_m$ is decidable and

$A \times A' \subseteq \bigcup_{(n,m)} (B_n \times B'_m)$. To verify the intersection property 2.5(1)(iii) in the definition of a space we use

$$(1) \qquad (B_{n_o} \times B'_{m_o}) \cap (B_{n_1} \times B'_{m_1}) \equiv (B_{n_o} \cap B_{n_1}) \times (B'_{m_o} \cap B'_{m_1})$$

for any n_o, n_1, m_o, m_1. Given (x,y) in the left-hand intersection, we find n, m with $x \in B_n \subseteq B_{n_o} \cap B_{n_1}$, and $y \in B_m \subseteq B'_{m_o} \cap B'_{m_1}$ to obtain $(x,y) \in (B_n \times B_m) \subseteq (B_{n_o} \times B'_{m_o}) \cap (B_{n_1} \times B'_{m_1})$. Thus $A \times A'$ _is a space_. It is also easily seen from (1) that $A \times A'$ inherits the property of being Hausdorff from that for each of A, A'. By induction one obtains for any n a Hausdorff space $A^n = \underbrace{A\times \ldots \times A}_{n}$.

Next we consider the _power_ (or _exponential_) spaces $A^{\mathbb{N}}$. As a structure this is given by the class of all sequences $\langle a_n \rangle$ from A, with equality of sequences $\langle a_n \rangle = \langle a'_n \rangle \Leftrightarrow \forall n[a_n = a'_n]$. Let s range over the class $\mathbb{N}^{<\omega}$ of finite sequences of natural numbers $s = \langle s_o, \ldots, s_{\ell h(s)-1} \rangle$. Associated with each of these is a set B^*_s in $A^{\mathbb{N}}$ defined by:

$$(2) \qquad \langle a_n \rangle \in B^*_s \Leftrightarrow \forall i < \ell h(s)(a_i \in B_{s_i}).$$

The relation $\langle a_n \rangle \in B^*_s$ is obviously decidable, and the sequence of all B^*_s is countable by an enumeration of $\mathbb{N}^{<\omega}$. Again it is easy to check the basic intersection property for this structure, so that $A^{\mathbb{N}}$ is verified to form a space. Finally, to show that being Hausdorff is inherited, given

$\langle a_n \rangle \neq \langle a'_n \rangle$, there exists n with $a_i = a'_i$ for $i < n$ and $a_n \neq a'_n$. Separating a_n, a'_n in A can then be used to form disjoint B^*_s, $B^*_{s'}$ with $\langle a_n \rangle \in B^*_s$ and $\langle a'_n \rangle \in B^*_{s'}$.

Of particular interest to us are <u>Baire space</u> $\mathbb{N}^{\mathbb{N}}$ with the basic sets

(3) $$[s] = \{f \in \mathbb{N}^{\mathbb{N}} \mid \bar{f}(\ell h(s)) = s\}$$

and its subspace <u>Cantor space</u> $2^{\mathbb{N}}$.

2.7 <u>Sequential compactness</u>. Compactness is one of the most important properties of a space (or suitable subspaces) used in theoretical analysis, so we shall go into it in some detail. Notions which are classically equivalent have to be reinvestigated in our weaker theories. In this section we shall treat sequential compactness, which is straightforward to deal with. In 2.15 below we shall take up compactness as formulated in terms of open coverings and which turns out to require more consideration of what axioms are assumed.

The space A is said to be <u>sequentially compact</u> if we can associate with every sequence $\langle x_n \rangle$ in $A^{\mathbb{N}}$ a convergent subsequence $\langle x_{n_k} \rangle$ $(n_0 < n_1 < \dots < n_k < \dots)$ <u>and</u> a limit x of this subsequence. This association is supposed to be given by a pair of functions $f\colon A^{\mathbb{N}} \to A^{\mathbb{N}}$, $f(\langle x_n \rangle) = \langle x_{n_k} \rangle$ and $g\colon A^{\mathbb{N}} \to A$, $g(\langle x_n \rangle) = \lim_k x_{n_k}$.

The discrete space $\mathbb{2} = \{0,1\}$ is seq. compact, by the following argument. Given a sequence $\langle x_n \rangle$ of 0's and 1's we can decide whether (i) $\exists n \forall m > n(x_m = 0)$ or (ii) $\forall n \exists m > n(x_m = 1)$. In the first case we take $\langle x_{n_k} \rangle = \langle x_n \rangle$, with limit 0. In the second case we let $\langle x_{n_k} \rangle$ be the subsequence of $\langle x_m \rangle$ consisting of all the 1's, i.e. $n_0 = \mu n(x_n = 1)$, $n_{k+1} = \mu n(n > n_m \wedge x_n = 1)$. This proof is in $VFT\restriction + (\mu)$.

We shall show in 2.10 below that for each $a, b \in \mathbb{R}$ with $a \leq b$, the subspace $[a,b] = \{x | a \leq x \leq b\}$ of $\mathbb{R}$ is seq. compact, again in $VFT\restriction + (\mu)$. Together with $\mathbb{2}$ these provide the basic examples of seq. compact spaces. We next consider closure of these spaces under products and powers.

2.8 <u>Sequential compactness for products and powers.</u> The situation with products is very simple:

(1) <u>If</u> A <u>and</u> B <u>are seq. compact then so also is</u> $A \times B$.

For, given $\langle (x_n, y_n) \rangle$ in $A \times B$, first choose a convergent subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ with limit x and then a convergent subsequence $\langle y_{n_{k_i}} \rangle$ of $\langle y_{n_k} \rangle$ with limit y so that $\langle (x_{n_{k_i}}, y_{n_{k_i}}) \rangle$ converges with limit (x,y). This proof is obviously given in $VFT\restriction + (\mu)$. It follows by induction outside of this theory that for each m we can prove

(2) <u>if</u> A <u>is seq. compact then so also is</u> A^m.

Consider the statement $S(A) \to \forall m \in \mathbb{N}(S(A^m))$ where $S(A)$ expresses that A is seq. compact. The property (of m) that $S(A^m)$ holds is not decidable and hence the proof of $S(A) \to \forall m \in \mathbb{N}(S(A^m))$ requires full induction. Thus

(3) (in VFT + (μ)) if A is seq. compact then so also is A^m for all m.

Next we turn to countable powers.

(4) $2^{\mathbb{N}}$ is seq. compact.

Here we shall use König's tree theorem for the argument. Suppose given a sequence $\langle f_n \rangle$ of elements of $2^{\mathbb{N}}$. Let b be the tree consisting of all sequences s of 0's and 1's such that $\forall n \exists m \geq n[\bar{f}_m(\ell h(s)) = s]$, i.e. such that infinitely many terms of the sequence $\langle f_n \rangle$ extend s. If s is in b then either $s * \langle 0 \rangle \in b$ or $s * \langle 1 \rangle \in b$. For if $s * \langle 0 \rangle \notin b$, there must be infinitely many f_n extending $s * \langle 1 \rangle$. By the tree theorem we can associate with b an infinite branch g through b. To complete the argument, $n_0 < n_1 < \ldots < n_k < \ldots$ are defined as follows:

$$n_0 = 0 \quad \text{and} \quad n_{k+1} = \mu n(n > n_k \text{ and } \overline{f_n}(k) = \bar{g}(k)); \quad \text{then} \quad \lim_k f_{n_k} = g.$$

This proof is given in VFT↾ + (μ). However, for the following stronger statement we again need stronger hypotheses.

(5) (In VFT + (μ)) if A is seq. compact then so also is $A^{\mathbb{N}}$.

For the proof (which is related to that for (1)) suppose given $\langle f_n \rangle$ where $f_n = \langle a_{n,m} \rangle_{m \in \mathbb{N}} \in A^{\mathbb{N}}$. Using the hypothesis on A we associate a sequence $n_0^{(0)} < n_1^{(0)} < \ldots < n_k^{(0)} < \ldots$ such that $\langle a_{n_k}^{(0)}, 0\rangle$ converges in A, say to a_0. Next associate a subsequence $n_0^{(1)} < n_1^{(1)} < \ldots < n_k^{(1)} < \ldots$ of $\langle n_k^{(0)}\rangle_{k\in\mathbb{N}}$ with $\langle a_{n_k}^{(1)}, 1\rangle$ convergent to an a_1 in A. Proceeding recursively (from sequences in $\mathbb{N}^{\mathbb{N}}$ to new such sequences) we obtain for each m a sequence $\langle n_k^{(m)}\rangle_{k\in\mathbb{N}}$ so that $\langle a_{n_k}^{(m)}, m\rangle$ is convergent to some a_n and $\langle n_k^{(m+1)}\rangle$ is a subsequence of $\langle n_k^{(m)}\rangle$. Then the sequence $\langle f_{n_k}(k)\rangle$ convergences in $A^{\mathbb{N}}$ to $\langle a_k\rangle$.

Exercise. Prove this last claim.

Remark. Note that it is essential for the proof of (5) to use $\varepsilon_{\mathbb{N}_1}$, with $\mathbb{N}_1 = (\mathbb{N} \rightarrow \mathbb{N})$, which is why $VFT\restriction + (\mu)$ does not suffice.

2.9 Topology of the real numbers. We continue with the real structure as defined in 1.8. Each real is a Cauchy sequence of rationals and the relation $<$ in reals is decidable by 1.8(3)(v), using quantification over $\mathbb{N}$, as is the relation $=$. For $a,b \in \mathbb{R}$ we put $[a,b] = \{x\in\mathbb{R} \mid a \leq x \leq b\}$, $[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$, $(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$ and $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$. Each of these intervals is a set. $\mathbb{R}$ is given the structure of a space by taking as basic open sets the rational open intervals (i.e. (a,b) with $a,b \in \mathbb{Q}$ and $a \leq b$) under a suitable enumeration. $\mathbb{R}$ is Hausdorff because if $x \neq y$ then $x < y$ or $y < x$. Say $x < y$ holds; then we can find $r, r', r'' \in \mathbb{Q}$ with $r < x < r' < y < r''$

and so $x \in (r,r')$, $y \in (r',r'')$ where $(r,r'),(r',r'')$ are disjoint. Again using density of $\mathbb{Q}$ in $\mathbb{R}$ we obtain that $\mathbb{R}$ is separable. Note that the intersection of any two rational open intervals is again such.

The first main result concerning $\mathbb{R}$ is that

(1) each closed interval $I = [a,b]$ is sequentially compact.

For the proof we use an argument by successive bisection which is essentially the same as that for seq. compactness of $2^{\mathbb{N}}$ by König's theorem. This is done first for the case that a,b are rational with $a < b$. Suppose given a sequence of reals $x_n \in I$; each x_n is a Cauchy sequence of rationals $x_n = \langle r_{n,m}\rangle_{m\in\mathbb{N}}$. With each finite sequence s of 0's and 1's, $s = \langle s_o,\ldots,s_{k-1}\rangle$ is associated a subinterval $[a_s,b_s]$ of I with $b_s - a_s = \frac{1}{2^k}(b-a)$. For $k=0$ take $a_s = a$, $b_s = b$. For $s' = s*\langle 0\rangle = \langle s_o,\ldots,s_{k-1}, 0\rangle$ we take $a_{s'} = a_s$ and $b_{s'} = \frac{1}{2}(a_s + b_s)$, i.e. $[a_{s'}, b_{s'}]$ is the left-most interval in the bisection of $[a_s,b_s]$. Similarly for $s' = s*\langle 1\rangle$ we take $a_{s'} = \frac{1}{2}(a_s+b_s)$ and $b_{s'} = b_s$. Next define a tree consisting of those s for which there are infinitely many n with $x_n \in [a_s,b_s]$. This tree is infinite since if s is in the tree, at least one of $s*\langle 0\rangle$, $s*\langle 1\rangle$ is in it. Thus we can find an infinite branch g through the tree; g has the property that there are for each k an infinity of n with x_n in $[a_{\bar{g}(k)}, b_{\bar{g}(k)}]$. Define $n_o = 0$ and $n_{k+1} = \mu n(n > n_k \wedge x_n \in [a_{\bar{g}(k+1)}, b_{\bar{g}(k+1)}])$; thus for all $n \geq k$,

$x_n \in [a_{\bar{g}(k)}, b_{\bar{g}(k)}]$. Finally, let $x = \langle a_{\bar{g}(k)} \rangle_{k \in \mathbb{N}}$, $x' = \langle b_{\bar{g}(k)} \rangle_{k \in \mathbb{N}}$. Each of x, x' is a Cauchy sequence of rationals and by construction $x = x'$. Any basic open neighborhood B_m of x contains $[a_{\bar{g}(k)}, b_{\bar{g}(k)}]$ for all sufficiently large k and hence contains x_{n_k} for all sufficiently large k. Thus $\lim_k x_{n_k} = x$, and we have associated with $\langle x_n \rangle$ both a convergent subsequence <u>and</u> the limit of that sequence. Now in general if a, b are not rational, we can find rational a', b' with $a' \leq a \leq b \leq b'$. Given any $\langle x_n \rangle$ in $[a,b]$, the preceding argument associates a convergent $\langle x_{n_k} \rangle$ with limit x in $[a',b']$. Since $[a,b]$ is closed it follows also that $x \in [a,b]$.

As a corollary we have that for each m,

(2) <u>each subspace of</u> $\mathbb{R}^m$ <u>of the form</u> $I_1 x \ldots x I_m$ <u>with</u> $I_j = [a_j, b_j]$ <u>is seq. compact</u>.

<u>Exercise</u>. Show that if $\bar{X}^{(s)} \subseteq X$ (i.e. X is closed under limits) and $X \subseteq A$ where A is seq. compact then also X is seq. compact.

2.10 <u>Metric spaces</u>. By a <u>metric</u> <u>space</u> we mean a set $\langle A, = \rangle$ together with a map $d : A^2 \to \mathbb{R}$ such that

(1) (i) $d(x,y) \geq 0$, $d(x,y) = 0 \Leftrightarrow x = y$,

(ii) $d(x,y) = d(y,x)$, and

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

In any metric space, the open spheres are defined to be the sets

(2) $S(x;a) = \{y \in A \mid d(x,y) < a\}$ for each $x \in A$ and $a \in \mathbb{R}$ (with $a \geq 0$).

A separable metric space is one for which we have a countable sequence $\langle q_n \rangle$ which is dense in A, i.e. such that

(3) for each $x \in A$ and $m > 0$ we can find n with $d(x,q_n) < \frac{1}{m}$.

Only separable metric spaces will be considered in the sequel. Fix an enumeration $\langle r_n \rangle$ of the rational numbers and an enumeration $n \mapsto (n_o, n_1)$ of $\mathbb{N}^2$, so that

(4) $$B_n = S(q_{n_o}; r_{n_1})$$

gives an enumeration of open spheres with center q_m and rational radius. It will be shown that A forms a topological space with countable basis in the sense of 2.4. Obviously the relation $x \in B_n$ is decidable. We use the following facts to prove 2.4(1):

(5) (i) Given $x \in A$ and $a \in \mathbb{R}$, $a > 0$, we can find n with $x \in B_n \subseteq S(x;a)$.

(ii) If $y \in S(x;a)$ and $d = a - d(x,y)$ then $S(y;d) \subseteq S(x;a)$.

(iii) Given $x_1, x_2 \in A$ and $a_1, a_2 \in \mathbb{R}$ with $a_1 > 0$, $a_2 > 0$ and $y \in S(x_1; a_1) \cap S(x_2; a_2)$ we can find n with $y \in B_n \subseteq S(x_1;a_1) \cap S(x_2;a_2)$.

For the proofs, first in (i) choose $m > 0$ with $\frac{1}{m} < \frac{a}{2}$ and $q_{n_o} \in S(x; \frac{1}{m})$; then we can take $B_n = S(q_{n_o}; \frac{1}{m})$. (ii) is immediate by the triangle inequality (1)(iii). For (iii), let $d = \min(a_1 - d(x_1,y),\ a_2 - d(x_2,y))$. Then $y \in S(y;d) \subseteq S(x_1;a_1) \cap S(x_2;a_2)$ by (ii), and we can find n such that $y \in B_n \subseteq S(y;d)$ by (i). The basic intersection property 2.4(1)(iii) of $\langle B_n \rangle$ is a special case of (4)(iii), while the covering property $A \subseteq \bigcup_n B_n$ of 2.4(1)(ii) follows immediately from (4)(i). In addition A is Hausdorff, because if $x \neq y$, then $d = d(x,y) > 0$, so $S(x;\frac{d}{2}) \cap S(y;\frac{d}{2})$ is empty. By (i) we can choose n,m with $x \in B_n \subseteq S(x,\frac{d}{2})$ and $y \in B_m \subseteq S(y;\frac{d}{2})$ so $B_n \cap B_m$ is empty.

Each $\mathbb{R}^m$ forms a metric space with the usual metric $d(x,y) = \sqrt{(x_1-y_1)^2 + \ldots + (x_m - y_m)^2}$ for $x = (x_1,\ldots,x_m)$, $y = (y_1,\ldots,y_m)$. (The existence of the square-root function for $\mathbb{R}$ follows from what has already been established in 2.9, as will be shown in §3.) For $n = 1$ the metric reduces simply to $d(x,y) = |x-y|$.

By a <u>Cauchy sequence</u> $\langle x_n \rangle$ in a metric space A we mean one for which

(6) $$\forall m > 0\ \exists k\ \forall n_1, n_2 \geq k\ \{d(x_{n_1}, x_{n_2}) < \tfrac{1}{m}\}.$$

A is said to be <u>complete</u> if every Cauchy sequence converges and we can find its limit. Note that

(7) $$\lim_n x_n = x \Leftrightarrow \forall m > 0\ \exists k\ \forall n \geq k\ \{d(x_n,x) < \tfrac{1}{m}\}.$$

Every Cauchy sequence is <u>bounded</u> since $\exists k\ \forall n_1, n_2 \geq k\ \{d(x_{n_1}, x_{n_2}) < 1\}$.

Let

(8) $$\bar{S}(x;a) = \{y \in A \mid d(y,x) \leq a\}$$

for each $x \in A$, $a \geq 0$. It is seen from (4)(i) that each $\bar{S}(x;a)$ is closed, so these are called <u>closed</u> <u>spheres</u> (or <u>balls</u>). Completeness of A follows from seq. compactness of each $\bar{S}(x;a)$. For, given Cauchy $\langle x_n \rangle$, since it is bounded we can find $a > 0$ such that each $x_n \in S(x_o\,;\,a)$. Then if $\langle x_{n_k} \rangle$ is a convergent subsequence and $\lim_k x_{n_k} = x$ we also have $\lim_n x_n = x$. In particular, we may use this observation to prove:

(9) <u>each</u> $\mathbb{R}^m$ <u>is</u> <u>complete</u>.

For, each $\bar{S}(x_o\,;\,a)$ is contained in a closed "cube" I^m for $I = [a_1, b_1]$, with suitable a_1, b_1. I^m was shown to be seq. compact in the preceding section. (Of course, R^m is separable with $q_n = (r_{n_1}, \ldots, r_{n_m})$ under an enumeration $n \mapsto (n_1, \ldots, n_m)$ of $\mathbb{N}^m$.

In a general separable metric space A we obtain the following simple description of the closure of open sets in A. Given $G = \bigcup_{n \in g} B_n$ where $g \in \mathbb{S}(\mathbb{N})$, suppose $x \in \bar{G}$, i.e. that for each $m > 0$ there exists $n \in g$ with $B_n \cap S(x\,;\frac{1}{m})$ non-empty. By (4)(iii) there exists non-empty B_p with $B_p \subseteq B_n \cap S(x;\frac{1}{m})$. Thus we have (i) of the following.

(10) (i) If $x \in \bar{G}$ where G is open then for each $m > 0$ we can find k with $q_k \in G$ and $d(q_k, x) < \frac{1}{m}$

(ii) $\bar{G} \equiv \bar{G}^{(s)}$.

The second part follows directly from (i), since we can find q_k as a function of x and m.

Exercise. Show that $\bar{S}(x;a) \equiv \overline{S(x;a)}$ and hence that each $\bar{S}(x;a)$ is closed.

2.11 The least upper bound property of $\mathbb{R}$. Let X be a non-empty subset of $\mathbb{R}$; a is said to be a supremum (sup) of X if for each $x \in X$, $x \leq a \wedge \forall b[\forall x \in X(x \leq b) \Rightarrow a \leq b]$. If such a exists for X then it is unique, so it is denoted $\sup X$ or $\sup_{x \in X} x$. Similarly for infimum of X, $\inf X$ and $\inf_{x \in X} x$. When X is the set $\{x_n\}$ of terms of a sequence, i.e. is countable, we write $\sup_n x_n$, $\inf_n x_n$ for these when they exist.

We know set-theoretically that the least upper bound property for $\mathbb{R}$ holds, i.e. if X is any non-empty set of reals which is bounded above then $\sup X$ exists. As we shall see in Ch.VI, this property cannot be established in $VFT + (\mu)$. The difficulty is that if we can prove existence of $a = \sup_{x \in X} x$ (for X bounded above), then we can obtain existence of a set b of rationals $r \geq a$. This set is characterized by

(4) $$r \in b \Leftrightarrow r \in \mathbb{Q} \wedge \forall x \in \mathbb{R}[x \in X \Rightarrow x \leq r].$$

In the defining property for b we use quantification over $\mathbb{R}$ in an essential way and thus implicitly over $\mathbb{N}^{\mathbb{N}}$. While in general sets definable in this way cannot be proved to exist in $VFT + (\mu)$ their existence follows directly from the further axiom $Proj_1$.

On the other hand we shall now show that

(5) <u>if</u> $\langle x_n \rangle$ <u>is a bounded sequence then</u> $\sup_n x_n$ <u>exists and can be found from</u> $\langle x_n \rangle$ (<u>and similarly for</u> $\inf_n x_n$).

For the proof define a monotone increasing subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ as follows: $n_o = 0$ and $n_{k+1} = \mu n(n > n_k$ and $x_{n_k} \leq x_n)$. This subsequence is finite if for a certain n_k, $x_n < x_{n_k}$ for all $n > n_k$, in which case $\sup_n x_n = x_{n_k}$. Otherwise we have $x_{n_o} \leq x_{n_1} \leq \ldots \leq x_{n_k} \leq \ldots$. In that case the result reduces to the following for monotonic sequences:

(6) <u>if</u> $x_o \leq x_1 \leq \ldots \leq x_n \leq \ldots \leq a$ <u>then</u> $\lim_n x_n$ <u>and</u> $\sup_n x_n$ <u>exist and are equal</u>.

For, by sequential compactness of $[x_o, a]$ there is a convergent subsequence of $\langle x_n \rangle$. Its' limit is easily seen to be a limit for the whole sequence and equal to $\sup_n x_n$. (A similar result holds for bounded monotone decreasing sequences.)

Set-theoretically the least upper bound property characterizes $\mathbb{R}$ (among ordered fields) and is the basic property of $\mathbb{R}$ which is constantly used in analysis. It must thus be shown in the pursuit of analysis in $VFT\!\upharpoonright + (\mu)$ that all such uses can be replaced by <u>least upper bound for sequences</u> $\sup_n x_n$ (<u>or greatest lower bound</u> $\inf_n x_n$). This will be examined in §§ 3 and 4.

We use (5) immediately for the description of open sets in $\mathbb{R}$. Fix an enumeration $r_n (n \in \mathbb{N})$ of the rationals and an enumeration $B_n = (r_{n_0}, r_{n_1})$ of rational open intervals with $r_{n_0} \leq r_{n_1}$. Each open set G is presented as $\bigcup_{n \in g} B_n$ for some $g \in \mathbb{S}(\mathbb{N})$. If G is non-empty we can drop empty intervals. From this we obtain a representation of G as a <u>disjoint union of open intervals</u> as follows. Put $n \equiv_1 m$ if $B_n \cap B_m$ is non-empty and $n \equiv m$ if there exist $p_0, \ldots, p_k$ with $p_0 = n$, $p_k = m$ and $p_0 \equiv_1 p_1 \equiv_1 \ldots \equiv_1 p_k$. This is an equivalence relation, and its equivalence classes $[n]$ are sets. For each n let $J_n = \bigcup_{m \in [n]} B_n$, and $a_n = \inf_{m \in [n]} (r_{m_0})$, $b_n = \sup_{m \in [n]} (r_{m_1})$. We use these inf's and sup's in the extended sense in $\mathbb{R} \cup \{\pm \infty\}$, e.g. if $\{r_{m_1} | m \in [n]\}$ is not bounded above, then $b_n = \infty$. Thus $J_n \equiv (a_n, b_n)$ and for any n, m, either $J_n \equiv J_m$ or $J_n \cap J_m$ is empty. By choosing inequivalent members of all the equivalence classes $n_0, \ldots, n_k, \ldots$ (of which there may only be a finite number), and taking $\bar{a}_k = a_{n_k}$, $\bar{b}_k = b_{n_k}$, we obtain:

(7) <u>with each non-empty open set</u> $G \equiv U(g)$ <u>of reals is associated a sequence</u> $\langle \bar{a}_k, \bar{b}_k \rangle$ <u>such that each</u> $\bar{a}_k < \bar{b}_k$ <u>and for</u> $k \neq \ell$, $(\bar{a}_k, \bar{b}_k) \cap (\bar{a}_\ell, \bar{b}_\ell)$ <u>is empty and</u> $G \equiv \bigcup_k (\bar{a}_k, \bar{b}_k)$.

This representation is obviously unique up to order.

2.12 <u>The complex numbers</u>. $\mathbb{C}$ is defined to consist of all pairs (a,b) of real numbers (where it is intended that (a,b) shall represent $a + bi$, $i = \sqrt{-1}$). The equality $=_{\mathbb{C}}$ is simply defined to be $=_{\mathbb{R} \times \mathbb{R}}$, i.e.

(1) $$(a_1,b_1) =_{\mathbb{C}} (a_2,b_2) \Leftrightarrow a_1 = a_2 \wedge b_1 = b_2$$

where on the r.h.s. we are using $=$ in the sense of $\mathbb{R}$. The algebraic operations on $\mathbb{C}$ are defined as follows:

(2) (i) $(a_1,b_1) + (a_2,b_2) = (a_1 + a_2, b_1 + b_2)$

(ii) $-(a,b) = (-a,-b)$

(iii) $(a_1,b_1)\cdot(a_2,b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)$

(iv) $$(a,b)^{-1} = \begin{cases} \left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right) & \text{if } a^2+b^2 \neq 0 \\ (0,0) & \text{if } a^2+b^2 = 0 . \end{cases}$$

It is a matter of routine verification to show that $\mathbb{C}$ <u>is a field</u>. We have an injection of $\mathbb{R}$ in $\mathbb{C}$ by $a \mapsto a_{\mathbb{C}} = (a,0)$. Let $i = (0,1)$. We shall identify a with $a_{\mathbb{C}}$ and $\mathbb{R}$ with its image under this map. Then $\mathbb{R}$ (without its ordering relation) is a subfield of $\mathbb{C}$, and $\mathbb{C}$ is generated from $\mathbb{R} \cup \{i\}$ simply by $(a,b) = (a,0) + (0,b) = (a,0) + (b,0)\cdot(0,1) = a+bi$. Next define

(3) (i) $|a+bi| = \sqrt{a^2+b^2}$ and

(ii) $d(z_1,z_2) = |z_1 - z_2|$.

Then $\mathbb{C}$ forms a metric space under d which is topologically the same as $\mathbb{R}^2$. Thus $\mathbb{C}$ is complete and each bounded disc $\{z \in \mathbb{C} \mid |z| \leq a\}$ is compact.

2.13 <u>The Baire category theorem</u>. We assume throughout this section that A is a separable metric space with dense subset $\langle q_n \rangle$. A set X is said to be <u>dense in</u> A if $\bar{X} \equiv A$. We shall consider in particular dense open sets in A, and use the following fact:

(1) If $G = \bigcup_{n \in g} B_n$ is dense in A then for each x in A and $r > 0$ we can find $n \in g$ and k with B_k non-empty and $\bar{B}_k \subseteq B_n \cap S(x; r)$.

For by the proof of (9) in 2.10, we can find non-empty $B_p \subseteq B_n \cap S(x; \frac{1}{m})$ where $\frac{1}{m} \leq r$; take B_k around the center of B_p and with smaller radins. Note for the following that it is decidable whether $\bar{B}_k \subseteq B_n \cap S(x; r)$.

The Baire category theorem can be established here in the following form.

(2) <u>Suppose A is complete and that $\langle g_m \rangle$ is a sequence of subsets of $\mathbb{N}$ such that each $G_m = U(g_m)$ is dense in A. Then $\bigcap_m G_m$ is non-empty; in fact it is also dense in A.</u>

To prove this, given any x_o and $r_o > 0$ we shall define a sequence $\langle k_m \rangle$ such that $\bar{B}_{k_{m+1}} \subseteq B_{n_m} \cap B_{k_m}$ for some n_m in g_m, and with the radius of $B_{k_m} \leq 1/m$. Begin with $x_o \in B_{k_o} \subseteq B(x_o; r_o)$. By (1), given k_m we can find $n_m \in g$ and k with $\bar{B}_k \subseteq B_{n_m} \cap B_{k_m}$; then k_{m+1} is chosen as the least such k for which the radius of B_k is $\leq 1/m$. (Note that only restricted recursion is involved here.) Let x_m be the center of

B_{k_m}. Since $x_{m'} \in B_{k_m}$ for all $m' \geq m$, the sequence $\langle x_m \rangle$ is Cauchy. Let $\lim_m x_m = x$. Then $x \in \bar{B}_{k_m}$ for each m so $x \in G_m$ for each m. (This uses only set-induction.) Given any x_o and $r_o > 0$ we have found $x \in S(x_o ; r_o)$ with $x \in \bigcap_m G_m$, so $\bigcap_m G_m$ is dense in A.

2.14 The contraction mapping theorem. This is an interesting example of a theorem which has a direct constructive proof in VFT but must be carefully modified in order to get a proof in VFT↾ + (μ). (The theorem itself permits one to obtain in a simple form some basic existence and uniqueness theorems for differential and integral equations.) Let A be a metric space. A function $f : A \to A$ is said to be a contraction mapping if for some r with $0 \leq r < 1$ we have

(1) $$d(f(x),f(y)) \leq r \cdot d(x,y) \quad \text{for all } x,y \in A.$$

The contraction mapping theorem is:

(2) *If A is complete and f is a contraction mapping on A then f has a unique fixed point.*

The proof in VFT runs as follows. Given any $x_o \in A$, define a sequence of points $\langle x_n \rangle$ by the recursion

(3) $$x_{n+1} = f(x_n).$$

It is seen that for any $n \geq 1$, $d(x_n, x_{m+1}) \leq r\, d(x_{n-1}, x_n) \leq r^2\, d(x_{n-2}, x_{n-1})$, etc., so

(4) $$d(x_n, x_{n+1}) \leq r^n \, d(x_o, x_1).$$

Hence

(5) $$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\ &\leq (r^n + r^{n+1} + \ldots + r^{n+k-1}) d(x_o, x_1) \\ &= r^n(1 + r + \ldots + r^{k-1}) d(x_o, x_1) \\ &= \frac{r^n r^k}{1-r} d(x_o, x_1) \leq \frac{r^n}{1-r} d(x_o, x_1). \end{aligned}$$

It follows that

(6) $\langle x_n \rangle$ is a Cauchy sequence, and

(7) for $x = \lim_n x_n$ we have $f(x) = x$.

The latter holds because $d(f(x), x_{n+1}) = d(f(x), f(x_n)) \leq r \cdot d(x, x_n)$, so $\lim_n d(f(x), x_n) = 0$ and $f(x) = \lim_n x_n = x$. Also

(8) if y is a fixed point of f then $x = y$,

since $d(x,y) = d(f(x), f(y)) \leq r \, d(x,y)$ which is not possible if $x \neq y$.

The preceding proof makes essential use of the general process of recursion $\underset{\sim}{r}_A$ in VFT ~~(I.3.4 Axiom VIII)~~ as well as induction on properties involving inequalities between reals; these properties do not define sets in VFT. To get a corresponding result using only restricted recursion $(r_{\mathbb{N}})$ and set-induction $(I_{\mathbb{N}})$ we pass to VFT$\Gamma + (\mu)$. The result to be proved is the same as (2) with the additional hypothesis that A <u>is now assumed to be separable</u>. Let $\langle q_n \rangle$ be dense in A. Thus:

(9) With each $x \in A$ and rational $\varepsilon > 0$ may be associated k such that $d(x, q_k) < \varepsilon$; this q_k is denoted $x^{(\varepsilon)}$.

Let r be as in (1) and fix r_1 and s with:

(10) $$r < r_1 < 1 \text{ and } s = \frac{r_1 - r}{2} .$$

We need the following lemma.

(11) For each $x, y \in A$ with $x \neq y$ and rational $\varepsilon > 0$ we can associate rational $\varepsilon' \leq \varepsilon$ such that $0 < \varepsilon' < s \cdot d(x^{(\varepsilon')}, y)$.

To prove (11), choose $0 < \varepsilon' \leq \min(\varepsilon, \frac{s}{1+s} d(x,y))$. Then $\varepsilon' + \varepsilon' s < sd(x,y)$ and $\varepsilon' < s[d(x,y) - \varepsilon'] < s \cdot d(x^{(\varepsilon')}, y)$, since $d(x,y) \leq d(x, x^{(\varepsilon')}) + d(x^{(\varepsilon')}, y) < \varepsilon' + d(x^{(\varepsilon')}, y)$.

Next in analogy to (3) define a sequence x_n as follows:

(12) (i) $x_0 = q_0$

(ii) $x_{n+1} = f(x_n)^{(\varepsilon_n)}$ where, if $f(x_n) \neq x_n$, the rational ε_n is chosen by (11) to satisfy $0 < \varepsilon_n \leq \frac{\varepsilon_{n-1}}{2}$ and $\varepsilon_n < s\, d(f(x_n)^{(\varepsilon_n)}, x_n)$.

By (9), each $x_n = q_{h(n)}$ for suitable h which is defined by the recursion process $\underset{\sim}{r}_{\mathbb{N}}$ along with the function $\lambda n \cdot \epsilon_n$ from $\mathbb{N}$ into $\mathbb{Q}$. If at any stage in (12) we get $f(x_n) = x_n$ we may stop, since this gives the desired conclusion. Hence we assume now that $f(x_n) \neq x_n$ for all n. If $n \geq 1$ we have

$$d(x_n, x_{n+1}) = d(f(x_{n-1})^{(\epsilon_{n-1})}, f(x_n)^{(\epsilon_n)})$$

$$\leq d(f(x_{n-1})^{(\epsilon_{n-1})}, f(x_{n-1})) + d(f(x_{n-1}), f(x_n)) + d(f(x_n), f(x_n)^{(\epsilon_n)})$$

$$< \epsilon_{n-1} + r \cdot d(x_{n-1}, x_n) + \epsilon_n .$$

Now $\epsilon_n \leq \epsilon_{n-1/2}$ so certainly $\epsilon_{n-1} + \epsilon_n \leq 2\, \epsilon_{n-1} < 2s\, d(f(x_{n-1})^{(\epsilon_{n-1})}, x_{n-1})$, i.e. $\epsilon_{n-1} + \epsilon_n < (r_1 - r) d(x_n, x_{n-1})$. It follows that

$$(13) \qquad d(x_n, x_{n+1}) < r_1 d(x_{n-1}, x_n).$$

We are thus able to conclude that $\langle x_n \rangle$ is a Cauchy sequence just as in (4)-(6). In this proof we still use inductive arguments on statements involving inequalities between reals. But such inequalities are decidable under the hypothesis (μ) and hence fall under set-induction.

To complete the argument, let $x = \lim_n x_n$. The proof that $f(x) = x$ is only slightly different from that for (7). Namely,

$$d(f(x), x_{m+1}) = d(f(x), f(x_n)^{(\epsilon_n)}) \leq d(f(x), f(x_n)) + d(f(x_n), f(x_n)^{(\epsilon_n)})$$

$$\leq r\, d(x, x_n) + \epsilon_n .$$

We have $\lim_n \epsilon_n = 0$ since $\epsilon_{n+1} \leq \epsilon_n/2$. Hence $\lim_n d(f(x),x_n) = 0$ and $f(x) = \lim_n x_n = x$.

Remark. The function (μ) was used in the second argument only to replace the induction scheme by $I_{\mathbb{N}}$. Otherwise this argument is still constructive.

2.15 Countable and full compactness. To complete the work of this section we see what can be said in VFT $+ (\mu)$ and its immediate extensions about compactness as formulated in terms of open coverings. We consider here any (Husdorff) topological space A with countable basis B_n. By a countable open cover of A we mean a sequence $\langle g_m \rangle$ of subsets of $\mathbb{N}$ for which $A \subseteq \bigcup_m G_m$ where $G_m = U(g_m)$; this is said to reduce to a finite subcover if for some k we have $A \subseteq \bigcup_{m \leq k} G_m$. By a full open cover of A we mean a set $a \subseteq \mathfrak{S}(\mathbb{N})$ for which $A \subseteq \bigcup_{g \in a} U(g)$; this is said to reduce to a finite subcover if for some $g_1,\ldots,g_k \in a$ we have $A \subseteq \bigcup_{m \leq k} U(g_m)$. A is said to be countably (fully) compact if every countable (full) open cover of A reduces to a finite subcover. There are dual formulations of these notions in terms of intersections and closed sets. For example, in the countable case it is that if $\langle f_n \rangle$ is any sequence of presentations of closed sets $F_n = U'(f_n)$ such that every finite sub-intersection is non-empty then $\bigcap_n F_n$ is non-empty. We shall say that A is strongly countably compact if we can associate with each sequence $\langle f_n \rangle$ satisfying the hypothesis an element $x \in \bigcap_n F_n$. This is equivalent to countable compactness if we assume the axiom of choice.

In studying the relations between these notions and seq. compactness we have to make use of a new property of spaces which is trivial if the axiom of choice is assumed. However, we shall show in 2.16 (next) that this property is held by all the concrete spaces which interest us. We say that A has the <u>cover witness (c.w.) property</u> if

(1) for each $n_0, \ldots, n_k$ we can associate $x \in A$ such that if $A \not\subseteq B_{n_0} \cup \ldots \cup B_{n_k}$ then $x \notin B_{n_0} \cup \ldots \cup B_{n_k}$.

Thus by examining whether or not the associated x belongs to $B_{n_0} \cup \ldots \cup B_{n_k}$, we can decide whether or not $A \subseteq B_{n_0} \cup \ldots \cup B_{n_k}$.

(2) <u>If A has the c.w. property and is seq. compact then A is strongly countably compact.</u>

For the proof of (2), suppose given a sequence $\langle f_n \rangle$ such that for $F_n = \bigcap_{m \in f_n} C_m$, each $F_0 \cap \ldots \cap F_n$ is non-empty. Let $f = \bigcup_{m \in f_n} f_n$ so that $\bigcap_n F_n \equiv \bigcap_{m \in f} C_m$. Enumerate f as $\{m_0, \ldots, m_k, \ldots\}$; then also each $C_{m_0} \cap \ldots \cap C_{m_k}$ is non-empty since it includes a finite intersection of the F_i's. Hence $A \not\subseteq B_{m_0} \cup \ldots \cup B_{m_k}$ for each k. By the cover witness property we can associate an $x_k \in A - (B_{m_0} \cup \ldots \cup B_{m_k})$. Then by seq.

compactness we associate a convergent subsequence $\langle x_{k_i}\rangle$ of the x_k's and an x such that $\lim_i x_{k_i} = x$. We have $x_{k_i} \in C_{m_0} \cap \ldots \cap C_{m_k}$ whenever $k_i \geq k$. Since each $C_{m_0} \cap \ldots \cap C_{m_k}$ is closed it is also closed under limits and we have $x \in C_{m_0} \cap \ldots \cap C_{m_k}$. Hence $x \in \bigcap_n F_n$.

(3) <u>If A is strongly countably compact then A is seq. compact.</u>

(Note that this does not require the c.w. property). For the proof, fix an enumeration $m \mapsto (m_0, m_1)$ of $\mathbb{N}^2$. Given any sequence $x_0, \ldots, x_k, \ldots$, we define

$$F_m = \begin{cases} C_{m_0} & \text{if } \forall k \geq m_1 (x_k \in C_{m_0}) \\ A & \text{otherwise} \end{cases} .$$

Thus $x \in F_m \Leftrightarrow [x \in C_{m_0} \vee \exists k \geq m_1 (x_k \in B_{m_0})] \Leftrightarrow [x \in B_{m_0} \Rightarrow \exists k \geq m_1 (x_k \in B_{m_0})]$. Each $F_0 \cap \ldots \cap F_m$ is non-empty since it contains x_k for sufficiently large k. Hence we can find $x \in \bigcap_m F_m$. Then $\forall n, m [x \in B_n \Rightarrow \exists k \geq m (x_k \in B_n)]$. To get a subsequence of the x_k which approaches x, enumerate all B_n containing x, say $B_{n_0}, \ldots, B_{n_i}, \ldots$. Then by the basic intersection property obtain a sequence $B_{n_0'} \supseteq \ldots \supseteq B_{n_i'} \supseteq \ldots$ with $x \in B_{n_i'} \subseteq B_{n_i}$. Finally define $k_0 = 0$, $k_{i+1} = \mu k (k > k_i \wedge x_k \in B_{n_i'})$; then $\lim_i x_{k_i} = x$.

<u>Remark</u>. Consider the set-theoretical statement: <u>countable compactness implies full compactness.</u> This has the following simple set-theoretical proof. Given any set $a \subseteq \mathcal{S}(\mathbb{N})$ which may be regarded as a set of open

presentations, let $g^* = \bigcup_{g \in a} g$. Then $\bigcup_{g \in a} U(g) \equiv \bigcup_{g \in a} \bigcup_{n \in g} B_n \equiv \bigcup_{n \in g^*} B_n$. Hence if $A \subseteq \bigcup_{g \in a} U(g)$, then $A \subseteq \bigcup_{n \in g^*} B_n$, so by countable compactness $A \subseteq B_{n_0} \cup \ldots \cup B_{n_k}$ for some $n_0, \ldots, n_k \in g^*$. Then also $A \subseteq U(g_0) \cup \ldots \cup U(g_k)$ for some $g_0, \ldots, g_k \in a$. This proof cannot be carried out in $VFT + (\mu)$ because existence of g^* requires definition by quantification over $\mathcal{S}(\mathbb{N})$. However, it will be shown in § 5 to follow very simply from the axiom $(Proj_1)$.

2.16 Verification of the cover witness property. The c.w. property is trivial for the space $\mathbf{2}$. To verify it for closed intervals $I = [a,b]$ in $\mathbb{R}$, consider any finite number of rational open intervals $(r_0, s_0), \ldots, (r_n, s_n)$, with $r_i < s_i$. The union of these is the same as that of a finite number of disjoint rational intervals, so we may assume such a representation from the beginning, placed in order with $s_i < r_{i+1}$. If $n > 0$ then this obviously does not cover $[a,b]$ and we can take $x = s_0$. If $n = 0$ then we take $x = r_0$ if $r_0 \geq a$ and otherwise take $x = s_0$.

Next we prove:

(1) If A, A' both have the c.w. property then so also does $A \times A'$.

Here we use the basic open sets $B_n \times B'_m$ and the fact

(2) $$(B_{n_0} \times B'_{m_0}) \cap (B_{n_1} \times B'_{m_1}) \equiv (B_{n_0} \cap B_{n_1}) \times (B_{m_0} \times B'_{m_1})$$

already mentioned in 2.6. Thus each basic open set of $A \times A'$ can be represented in the form

(3) $$(B_n \times B'_m) \equiv (B_n \times A') \cap (A \times B'_m),$$

and consequently

(4) $$(A \times A') - (B_n \times B'_m) \equiv [-(B_n \times A')] \cup [-(A \times B'_m)] \equiv (C_n \times A') \cup (A \times C'_m).$$

Consider any sequence $B_{n_o} \times B'_{m_o}, \ldots, B_{n_k} \times B'_{m_k}$. We have $A \times A' \nsubseteq (B_{n_o} \times B'_{m_o}) \cup \ldots \cup (B_{n_k} \times B'_{m_k})$ just in case $[-(B_{n_o} \times B'_{m_o})] \cap \ldots \cap [-(B_{n_k} \times B'_{m_k})]$ is non-empty, i.e. just in case

(5) $[(C_{n_o} \times A') \cup (A \times C'_{m_o})] \cap \ldots \cap [(C_{n_k} \times A') \cup (A \times C'_{m_k})]$ is non-empty.

By the distributive law, (5) is equivalent to the assertion:

(6) for some finite sequence $s_o, \ldots, s_k$ of 0's and 1's we have

$$\bigcap_{i \leq k,\ s_i = 0} (C_{n_i} \times A') \cap \bigcap_{i \leq k,\ s_i = 1} (A \times C_{m_i}) \neq \Lambda .$$

But the intersection in (6) is the same as $[(\bigcap_{i \leq k, s_i = 0} C_{n_i}) \times A'] \cap [A \times \bigcap_{i \leq k, s_i = 1} C'_{m_i}]$.

Thus (6) is equivalent to the assertion that for some binary sequence $s_o, \ldots, s_k$ we have $\bigcap_{i \leq k,\ s_i = 0} C_i \neq \Lambda$ and $\bigcap_{i \leq k, s_i = 1} C'_i \neq \Lambda$. By the cover-witness property for A, A' resp. we can find for each such s a pair x_s, y_s such that $x_s \in \bigcap_{i \leq k, s_i = 0} C_i$ if it is non-empty and $y_s \in \bigcap_{i \leq k, s_i = 1} C'_i$ if it is non-empty. Thus (x_s, y_s) belongs to the intersection in (6) if it is non-empty. Finally define (x,y) as follows: in an enumeration of all binary $s = \langle s_0, \ldots, s_k \rangle$ take the first (x_s, y_s) which belongs to the intersection in (6) if any, otherwise fix (x,y) arbitrarily. This proves (1).

It follows by induction that

(7) <u>if</u> A <u>has the c.w. property then so does</u> A^m <u>for each</u> m.

Finally, for the power spaces we have

(8) <u>if</u> A <u>has the c.w. property then so does</u> $A^{\mathbb{N}}$.

To establish this, consider any finite number of basis sets $B^*_{s^{(0)}}, \ldots, B^*_{s^{(k)}}$, where each $s^{(i)}$ is a finite sequence of elements of A, say of length $m^{(i)}$. Let $n = \max_i m^{(i)}$. Then $B^*_{s^{(i)}}$ acts like $B_{s_0^{(i)}} \times \ldots \times B_{s_{m_i - 1}^{(i)}} \times A^{m - m_i}$ in A^m and we can obtain the appropriate cover-witness by the method for A^m.

As a corollary to the results of 2.9, 2.10, the preceding section and this section, we have:

(9) <u>Each</u> $[a_1,b_1] \times \dots \times [a_m,b_m] (\subseteq \mathbb{R}^m)$ <u>is strongly countably compact; the same holds for</u> $2^{\mathbb{N}}$.

3. <u>Developments in</u> VFT↾ + (μ): <u>Classical analysis</u>. Throughout this section, A is assumed to be a separable metric space with dense basis $\langle q_n \rangle$. We fix an enumeration $B_n = S(q_{n_0}; r_{n_1})$ of the basic open spheres in A as in 2.10. Additional hypotheses which may be placed on A are: completeness, countable compactness and sequential compactness. Basic notions of continuous and of uniformly continuous function $f: A \to A'$ are examined in this general setting (where A' is also separable metric, with basis $\langle q'_n \rangle$ and basic opens $B'_n = S(q'_{n_0}; r_{n_1})$). The essential point is to strengthen the usual definition so that we are provided with a <u>modulus-of-continuity function</u> $\delta = \delta(\epsilon)$. As is to be expected, continuity implies uniform continuity on suitably compact A; also maxima and minima are attained. The intermediate value theorem for f in $C(A,\mathbb{R})$ is established for connected A, where $C(A,\mathbb{R})$ is the class of all continuous $f: A \to \mathbb{R}$.

The next general topic concerns convergence and uniform convergence of sequences and series in $C(A,\mathbb{R})$. When A is seq. compact the norm $\|f-g\| = \sup_{x \in A} |f(x)-g(x)|$ is defined, with respect to which $C(A,\mathbb{R})$ forms a metric space. The Stone-Weierstrass theorem is proved for countably generated algebras; when we have such, the space $C(A,\mathbb{R})$ is separable.

(The classical Weierstrass approximation theorem is a corollary.)

After this general work we shift to a sketch of the differential and integral calculus on $\mathbb{R}$, the latter via Riemann integration. Sets of measure 0 are introduced to characterize the Riemann integrable functions. Power series are used to obtain a large stock of classical functions. We conclude with a brief tour of the topics of existence theorems for differential equations, Fourier series representation, and complex analysis.

3.1 Continuity. For simplicity, throughout ϵ, δ range over $\mathbb{Q}^+$ (the set of positive rational numbers). A function $f : A \to A'$ is said to be continuous on $A_0 \subseteq A$ if we have a function $\delta = \delta(x, \epsilon)$ (on $A \times \mathbb{Q}^+$) such that

(1) $$x \in A_0 \wedge y \in S(x; \delta(x, \epsilon)) \Rightarrow f(y) \in S(f(x); \epsilon).$$

If this holds with $A_0 = \{x\}$ we say f is continuous at x, and when $A_0 = A$ we simply say that it is continuous. $C(A, A')$ denotes the class of all f which are continuous. It is easily seen that

(2) if f is continuous at x and $x = \lim_n x_n$ then $f(x) = \lim_n f(x_n)$.

The converse cannot be derived without the axiom of choice.

There is another familiar definition of continuity (on A) in terms of inverse images of open sets. Given $X \subseteq A$, $Y \subseteq A'$ write

$f[X] = \{f(x) | x \in X\}$ and $f^{-1}[Y] = \{x \in A | f(x) \in Y\}$. We say that f is O-continuous if the f-inverse image of each open set in A' is an open set in A, in the strong sense that for each $g \in \mathcal{S}(\mathbb{N})$ we can find $\tilde{g} \in \mathcal{S}(\mathbb{N})$ such that

$$(3) \qquad f^{-1}[\bigcup_{m \in g} B'_m] \equiv \bigcup_{n \in \tilde{g}} B_n .$$

In the remainder of this section we prove the equivalence of O-continuity with continuity.

First of all suppose f is O-continuous. To show that it is continuous, consider any $x \in A$ and $\epsilon > 0$; a number $\delta = \delta(x;\epsilon)$ will be defined to satisfy (1). Represent the open set $S(f(x);\epsilon)$ in A' as $\bigcup_{n \in g} B'_n$; g is the set of n such that $S(q'_{n_0}; r_{n_1}) \subseteq S(f(x);\epsilon)$, i.e. the set of n such that $d(f(x), q_{n_0}) + r_{n_1} \leq \epsilon$ (which is decidable). By O-continuity, we can find $\tilde{g}$ satisfying $f^{-1}[S(f(x);\epsilon)] \equiv \bigcup_{m \in \tilde{g}} B_m$. Since x belongs to this set we can find $m \in \tilde{g}$ such that $x \in B_m$; we may then determine δ such that $S(x;\delta) \subseteq B_m$, as desired.

For the converse, suppose f is continuous. First note that we can decide whether or not $f[S(q_m;\delta)] \subseteq \bar{S}(q'_n;\epsilon)$ (the closure of $S(q'_n;\epsilon)$). Namely, this is equivalent by (2) to $f(q_i) \in \bar{S}(q'_n;\epsilon)$ for each $q_i \in S(q_m;\delta)$. Now given any g rewrite $\bigcup_{n \in g} S(q'_{n_0}; r_{n_1})$ as $\bigcup_{n \in g,\, k \in \mathbb{N}} \bar{S}(q'_{n_0}; (1-\frac{1}{2^{k+1}})r_{n_1})$, and let $\tilde{g}$ be the set of m such that $f[S(q_{m_0}; r_{m_1})] \subseteq \bar{S}(q'_{n_0}; (1-\frac{1}{2^{k+1}})r_{n_1})$

for some k,n. This choice satisfies (3). For, $f[\bigcup_{m\in \tilde{g}} B_m] \subseteq \bigcup_{n\in g} B'_n$ by construction, so $\bigcup_{m\in \tilde{g}} B_m \subseteq f^{-1}[\bigcup_{n\in g} B'_n]$. To prove the reverse, suppose $x \in f^{-1}[\bigcup_{n\in g} B'_n]$, so $f(x) \in B'_n$ for some n. Then we can find k so that $f(x) \in S(q'_{n_0}; (1-\frac{1}{2^{k+1}})r_{n_1})$ and from that a δ so that $f[S(x;\delta)] \subseteq S(q'_{n_0}; (1-\frac{1}{2^{k+1}})r_{n_1})$. The proof is completed by choosing m such that $x \in B_m \subseteq S(x;\delta)$.

3.2 <u>Uniform continuity</u>. $f: A \to A'$ is defined to be <u>uniformly continuous</u> if we have a function $\delta = \delta(\epsilon)$ such that

(1) for any x in A, $f[S(x;\delta(\epsilon))] \subseteq S(f(x);\epsilon)$.

The main result here is that

(2) <u>if A is countably compact and f is continuous then it is uniformly continuous</u>.

The proof is in two steps. In the first step we prove uniform continuity on $\{q_n\}_{n\in \mathbb{N}}$. Fix any $\epsilon > 0$. Let g be the set of all n such that $\forall_i[q_i \in S(q_{n_0}; 2r_{n_1}) \Rightarrow f(q_i) \in S(f(q_{n_0}); \frac{\epsilon}{2})]$. Given any $x \in A$ there exists $n \in g$ with $x \in B_n$. For first we can find a δ (from x, ϵ) such that $f[S(x;\delta)] \subseteq S(f(x); \frac{\epsilon}{4})$ and then choose n with $x \in B_n \subseteq S(q_{n_0}; 2r_{n_1}) \subseteq S(x;\delta)$. Hence if $q_i \in S(q_{n_0}; 2r_{n_1})$ we have

$d(f(q_i), f(q_{n_o})) \leq d(f(q_i), f(x)) + d(f(x), f(q_{n_o})) < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$.

It follows that $A \subseteq \bigcup_{n \epsilon g} B_n$. Hence by countable compactness there exists finite $g_o \subseteq g$ with $A \subseteq \bigcup_{n \epsilon g_o} B_n$. Let δ_1 be the minimum of the radii of the B_n's for $n \epsilon g_o$. We have:

(3) $$d(q_i , q_j) < \delta_1 \Rightarrow d(f(q_i), f(q_j)) < \epsilon .$$

For let $q_i \epsilon B_n = S(q_{n_o} ; r_{n_1})$ for $n \epsilon g_o$; by $d(q_i,q_j) < \delta \leq r_{n_1}$ we have $q_j \epsilon S(q_{n_o} ; 2 r_{n_1})$. Then both $f(q_i)$, $f(q_j)$ belong to $S(f(q_{n_o}); \frac{\epsilon}{2})$ by construction, from which we get (3). We denote by $\delta_1(\epsilon)$ the function given by this proof to satisfy (3).

To complete the proof, let $\delta = \delta(\epsilon) = \frac{1}{3} \delta_1(\frac{\epsilon}{3})$. Suppose $d(x,y) < \delta$. By continuity we can find $\delta_2 = \delta_2(x, \epsilon)$, $\delta_3 = \delta_3(y, \epsilon)$ so that $f[S(x; \delta_2)] \subseteq S(f(x); \frac{\epsilon}{3})$ and $f[S(y; \delta_3)] \subseteq S(f(y); \frac{\epsilon}{3})$. These may also be chosen with $\delta_2 \leq \delta$, $\delta_3 \leq \delta$. Now we can find q_i , q_j with $d(q_i,x) < \delta_2$ and $\delta(q_j,y) < \delta_3$. Hence

$$d(q_i,q_j) \leq d(q_i,x) + d(x,y) + d(y,q_j) < 3\delta = \delta_1(\frac{\epsilon}{3})$$

and then $d(f(q_i), f(q_j)) < \frac{\epsilon}{3}$ by (3). Finally

$$d(f(x), f(y)) \leq d(f(x), f(q_i)) + d(f(q_i), f(q_j)) + d(f(q_j), f(y)) < 3(\frac{\epsilon}{3}) = \epsilon .$$

Thus $\delta(\epsilon)$ satisfies (1) as required.

3.3 Attainment of maxima and minima. Suppose $f : A \to \mathbb{R}$; f is defined to attain a maximum at $x \in A$ if $f(x) \geq f(y)$ for all $y \in A$ (similarly for minima). We prove:

(1) if f is continuous and bounded above (below) and A is seq. compact then f attains a maximum (minimum) on A.

First of all we know that $M = \sup_n f(q_n)$ exists by 2.11. Since f is continuous also $f(y) \leq M$ for each $y \in A$ and hence $M = \sup_{y \in A} f(y)$. Now for each k choose n_k so that $M - f(q_{n_k}) < \frac{1}{k}$. Then choose a convergent subsequence of $\langle q_{n_k} \rangle$ and let x be its limit. We have $f(x) = M$.

3.4 Connected spaces and the intermediate value theorem. The space A is said to be connected if there is no X with X and $A-X$ both non-empty and open. (If there is such X then it is both open and closed). We have the following result.

(1) If $f : A \to A'$ is continuous and onto and A is connected then A' is connected.

For the proof, suppose that there exists $Y \subseteq A'$ such that Y and $A' - Y$ are both non-empty and open. Then by 3.1, $X = f^{-1}(Y)$ and $A' - X = f^{-1}(A'-Y)$ are both non-empty and open, contradicting the hypothesis.

To apply (1) we characterize the connected subspaces A of $\mathbb{R}$. A is called an <u>interval</u> if $a,b \in A$ with $a < b$ implies $[a,b] \subseteq A$. Besides the finite intervals from 2.9 these also comprise the infinite intervals (a, ∞), $[a, \infty)$, $(-\infty, b)$ $(-\infty, b]$ and $(-\infty, \infty)$.

(2) <u>Every interval</u> A <u>in</u> $\mathbb{R}$ <u>is connected</u>.

For suppose given $X_1 \subseteq A$ with both X_1 and $A - X_1$ non-empty and open, so $X_1 = A \cap X$ for an open X in $\mathbb{R}$. Let $a \in X$, $b \in A - X$ and say $a < b$; thus $X_1 \cap [a,b] = X \cap [a,b]$. Represent X as a union of disjoint intervals $X = \bigcup_n (a_n, b_n)$; we may drop those (a_n, b_n) with $b \leq a_n$. Then $\sup X = \sup_n b_n \leq b$ and $\sup X \notin X$. Since $[a,b] - X$ is supposed to be open in $[a,b]$ there must be an $\epsilon > 0$ such that $(\sup X - \epsilon, \sup X) \subseteq [a,b] - X$. However, this contradicts the definition of sup.

Conversely we have the following:

(3) <u>if a subspace</u> A <u>of</u> $\mathbb{R}$ <u>is connected then</u> A <u>is an interval</u>.

The proof is trivial; if $a,b \in A$ with $a < b$ and $c \in [a,b] - A$ then $(-\infty, c) \cap A$ and $(c, \infty) \cap A$ are both non-empty, open and complementary (relative to A).

We draw as a corollary from (1) and (3) the Intermediate Value Theorem:

(4) <u>if</u> $f: A \to \mathbb{R}$ <u>is continuous and</u> A <u>is connected then</u> $f[A]$ <u>is an interval</u>.

In particular, by (2) if A is an interval in $\mathbb{R}$ then $f[A]$ is also an interval. Hence, if f is continuous on $[a,b]$ and y lies between $f(a)$ and $f(b)$ there exists some $x \in [a,b]$ with $f(x) = y$. In fact, a particular such x can always be found as follows: for each $n > 0$, find a rational $x_n \in [a,b]$ with $|y - f(x_n)| < \frac{1}{n}$; then take x to be $\lim_k x_{n_k}$ where $\langle x_{n_k} \rangle$ is a convergent subsequence of $\langle x_n \rangle$.

As an example, let $f(x) = x^2$ and suppose $0 < y$. Take $a = 0$ and $b > y^2$; then we can find x in $[a,b]$ with $x^2 = y$. Such x is unique and is denoted $\sqrt{y}$. In this way we obtain the square-root function needed to define the metric on $\mathbb{R}^m$ for $m > 1$ (as remarked in 2.10).

3.5 <u>Sequences and series</u>. Given a sequence of reals $\langle x_n \rangle$ we form the sequence $s_n = \Sigma_{k=0}^{n} x_k$ of partial sums in an elementary way as follows. Given n, s_n is a Cauchy sequence of rationals whose ith term is $\Sigma_{k=0}^{n} x_k(i)$. By this means we avoid use of the recursion operator $\underset{\sim}{r}_{\mathbb{R}}$ replacing it by $\underset{\sim}{r}_{\mathbb{N}}$. Now $\Sigma_{k=0}^{\infty} x_k$ (or $\Sigma_k x_k$) is said to be <u>convergent</u> if its sequence of partial sums $\langle s_n \rangle$ is convergent. The value of the series, when convergent is defined by

(1) $$\Sigma_{k=0}^{\infty} x_k = \lim_n (\Sigma_{k=0}^{n} x_k).$$

The Cauchy criterion for convergence of sequences immediately transfers to series by:

(2) $\Sigma_k x_k$ <u>converges iff for each</u> $\epsilon > 0$ <u>we can find</u> k <u>such that</u>

$$\forall n_2 \geq n_1 \geq k \{ | \Sigma_{k=n_1}^{n_2} x_k | < \epsilon \}.$$

In particular, it is seen that if $\Sigma_k x_k$ converges then $\lim_k x_k = 0$. Further, by $|\Sigma_{k=n_1}^{n_2} x_k| \leq \Sigma_{k=n_1}^{n_2} |x_k| = |\Sigma_{k=1}^{n_2} |x_k||$, we see that

(3) <u>if</u> $\Sigma_k x_k$ <u>converges absolutely then it converges</u>,

where the hypothesis means that $\Sigma_k |x_k|$ converges.

We next consider sequences $\langle f_n \rangle$ and series $\Sigma_n f_n$ of functions $f_n : \mathbb{R} \to \mathbb{R}$. Here $\langle f_n \rangle$ is supposed to be given by a function

$g : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ with $f_n(x) = g(n,x)$. Convergence of $\langle f_n(x) \rangle$ and $\Sigma_n f_n(x)$ will vary from one value of x to another. They thus define in the limit functions with domain a subset A of $\mathbb{R}$ $\lim_n f_n(x)$ and $\Sigma_{n=0}^{\infty} f_n(x)$. A common problem concerns when we can extend properties such as continuity from each f_n to the limit or sum function. For example, each of the functions $f_n(x) = x^n$ is continuous but the limit function is not. A basic concept is that of <u>uniform convergence of</u> $\langle f_n \rangle$ <u>to</u> f <u>on</u> A which is defined to hold if for each $\epsilon > 0$ we can find k such that

(4) <u>for all</u> $n \geq k$ <u>and all</u> $x \in A$, $|f_n(x) - f(x)| < \epsilon$.

In other words, the choice of k is independent of x. The series $\Sigma_k f_k$ is called uniformly convergent on A if its sequence of partial sums has this property. We have a form of Cauchy criterion for sequences of functions:

(5) $\langle f_n \rangle$ <u>converges uniformly</u> (<u>to some</u> f) <u>on</u> A <u>iff for each</u> $\epsilon > 0$

$$\exists k \; \forall n_1, n_2 \geq k \;\; \forall x \in A \; \{|f_{n_1}(x) - f_{n_2}(x)| < \epsilon\} .$$

For the proof, suppose $\langle f_n \rangle$ converges uniformly to f on A. Given ϵ pick k such that $\forall n \geq k \; \forall x \in A \{|f_n(x) - f(x)| < \epsilon/2\}$. Then $\forall n_1, n_2 \geq k \; \forall x \in A \{|f_{n_1}(x) - f_{n_2}(x)| < \epsilon\}$. Conversely, if the condition holds then $\langle f_n(x) \rangle$ is a Cauchy sequence for each $x \in A$ and hence $\lim_n f_n(x) = f(x)$ is defined on A. It is easily verified that $\langle f_n \rangle$ converges uniformly to f on A. A convenient sufficient test for uniform convergence of series is the Weierstrass M-test:

(6) if $\Sigma_n M_n$ converges and $|f_n(x)| \leq M_n$ for each $x \in A$ then $\Sigma_n f_n$ converges uniformly on A.

The proof is immediate by definition.

The main result here is that

(7) if $\langle f_n \rangle$ converges uniformly on A and each f_n is continuous on A with modulus-of-continuity function $\delta_n = \delta_n(x;\epsilon)$ then f is continuous on A.

To prove this, given $\epsilon > 0$, choose n so that $|f_n(y) - f(y)| < \epsilon$ for all $y \in A$. Given any $x \in A$, take $\delta(x;\epsilon) = \delta_n(x;\epsilon/3)$. Then for $|y-x| < \delta(x;\epsilon)$ we have

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| < \epsilon .$$

It is clear from the proof that if each f_n is uniformly continuous with modulus-of-uniform continuity function $\delta_n(\epsilon)$ then f is uniformly continuous using $\delta(\epsilon) = \delta_n(\epsilon/3)$ for n as described.

3.6 The space $C(A,\mathbb{R})$. We assume here that A is strongly countably compact, hence seq. compact by 2.15. Each f in $C(A,\mathbb{R})$ is uniformly continuous by 3.2, so there exists a function $\delta = \delta(\epsilon)$ giving its uniform modulus of continuity:

$$(1) \qquad x,y \in A \wedge d(x,y) < \delta(\epsilon) \Rightarrow |f(x) - f(y)| < \epsilon .$$

When operating with members of $C(A,\mathbb{R})$ we must make use of these associated functions. Thus we here identify $C(A,\mathbb{R})$ with the class of all pairs (f,δ) satisfying (1). However for simplicity of notation we continue to denote elements of $C(A,\mathbb{R})$ by their first terms where there is no ambiguity.

For $f,g \in C(A,\mathbb{R})$ we obtain as usual $|f|$, $f+g$, $f-g$, $f \cdot g$ in $C(A,\mathbb{R})$; further $f/g \in C(A,\mathbb{R})$ when $g \neq 0$ (i.e. $\forall x \in A(g(x) \neq 0)$). We also have $\max(f,g)$ and $\min(f,g)$ in $C(A,\mathbb{R})$, using $\max(f,g) = \frac{(f+g)}{2} + \frac{|f-g|}{2}$ and similarly for min. We also define

$$(2) \qquad \|f\| = \sup_{x \in A} |f(x)| ,$$

which exists by 3.3 (and, indeed, satisfies $\|f\| = |f(a)|$ for some $a \in A$). Then it is easily seen that

(3) $C(A,\mathbb{R})$ forms a metric space under the function $d(f,g) = \|f-g\|$.

A sequence of elements $\langle f_n \rangle$ of $C(A,\mathbb{R})$ is actually given by a pair of sequences, $(\langle f_n \rangle, \langle \delta_n \rangle)$ where $\delta_n = \delta_n(\epsilon)$ is a uniform modulus of continuity function for f_n. From 3.5 we obtain:

(4) (i) $\langle f_n \rangle$ <u>converges uniformly to</u> $f \Leftrightarrow \lim_n \|f_n - f\| = 0$;

(ii) $\langle f_n \rangle$ <u>is uniformly convergent</u> $\Leftrightarrow \langle f_n \rangle$ <u>is a Cauchy sequence in</u> $C(A,\mathbb{R})$.

The main new fact here is that

(5) $C(A,\mathbb{R})$ <u>is complete</u>.

For the proof, suppose given a Cauchy sequence $\langle f_n \rangle$ with accompanying moduli δ_n. Let $f = \lim_n f_n$. We must show that $f \in C(A,\mathbb{R})$ by finding a function δ satisfying (1). By hypothesis we have a function $n = n(\epsilon)$ such that for each $\epsilon > 0$, $[m \geq n(\epsilon) \wedge x \in A \Rightarrow |f_m(x) - f(x)| < \epsilon]$. Also for each n we have that $[x,y \in A \wedge d(x,y) < \delta_n(\epsilon) \Rightarrow |f_n(x) - f_n(y)| < \epsilon]$. Let $m = n(\frac{\epsilon}{3})$ and $\delta = \delta_m(\epsilon/3)$. Then

$$|f(x) - f(y)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f(y)| ,$$

which is less than ϵ whenever $d(x,y) < \delta$.

As the example $f_n(x) = x^n$ on $[0,1]$ shows, ordinary convergence does not imply uniform convergence. However, we have Dini's theorem :

(6) <u>if</u> $\langle f_n \rangle$ <u>is a sequence in</u> $C(A,\mathbb{R})$ <u>and</u> $(\lim_n f_n) \in C(A,\mathbb{R})$ <u>and</u> $f_0 \leq f_1 \leq \dots \leq f_n \leq \dots$ <u>then</u> $\langle f_n \rangle$ <u>converges uniformly</u>.

The proof is left to the reader.

It follows of course from (5) that

(7) <u>if</u> $\langle f_n \rangle$ <u>is a sequence in</u> $C(A, \mathbb{R})$ <u>and</u> $\sum_{n=0}^{\infty} f_n$ <u>converges uniformly then</u> $\sum_{n=0}^{\infty} f_n \in C(A, \mathbb{R})$.

<u>Exercise</u>. Prove Dini's theorem. (Outline: let $f = \lim_n f_n$. Given ϵ, n, m, let $(n,m) \in g \Leftrightarrow \forall q_i \in B_n[(f(q_i) - f_m(q_i)) \leq \epsilon] \Leftrightarrow \forall y \in B_n[(f(y) - f_m(y)) \leq \epsilon]$. Let $G_{n,m} = B_n$ for $(n,m) \in g$. Then the $G_{n,m}$ with $(u,m) \in g$ form an open cover of A.)

3.7 <u>The Stone-Weierstrass Theorem</u>. We continue to assume that A is strongly countably compact. The Stone-Weierstrass theorem will establish that $C(A, \mathbb{R})$ is separable when a certain hypothesis is met; this is easily verified for the A in which we are interested. We denote by c the constant function $\lambda x \in A(c)$ for $c \in \mathbb{R}$. A family $\mathfrak{a} \subseteq C(A, \mathbb{R})$ is said to be an <u>algebra</u> if

(1) $f, g \in \mathfrak{a} \Rightarrow f + g \in \mathfrak{a}$, $f \cdot g \in \mathfrak{a}$ and $cf \in \mathfrak{a}$ for each $c \in \mathbb{Q}$.

$\mathfrak{a}$ is said to <u>separate points</u> if for any $x_1, x_2 \in A$ with $x_1 \neq x_2$ and any $a_1, a_2 \in \mathbb{R}$ we can associate a function $f \in \mathfrak{a}$ with

(2) $$f(x_1) = a_1 \wedge f(x_2) = a_2 .$$

$\mathfrak{a}$ is said to be <u>uniformly closed</u> if it is closed under uniformly convergent sequences, i.e. under limits in the space $C(A, \mathbb{R})$. The <u>uniform closure of</u>

$\mathcal{A}$ is $\overline{\mathcal{A}}^{(s)}$.

The Stone-Weierstrass theorem is here stated as follows.

(3) Suppose $\mathcal{A}$ is a countable algebra and $\mathcal{B} = \overline{\mathcal{A}}^{(s)}$. If $1 \in \mathcal{B}$ and $\mathcal{B}$ separates points then $\mathcal{B} \equiv C(A, \mathbb{R})$.

For the proof, first of all it is easily checked that

(4) $\mathcal{B}$ is a uniformly closed algebra.

Let $\mathcal{A} = \{p_n\}_{n \in \mathbb{N}}$. By definition of $\mathcal{B}$, we have

(5) for each $f \in \mathcal{B}$ and $\epsilon > 0$ there exists $p_n \in \mathcal{A}$ with $\|f - p_n\| < \epsilon$.

Thus $\mathcal{A}$ separates points approximately, in the sense that for any $x_1 \neq x_2$ and $a_1, a_2 \in \mathbb{R}$ we can find a $p_n \in \mathcal{A}$ with

(6) $$|p_n(x_1) - a_1| < \epsilon \wedge |p_n(x_2) - a_2| < \epsilon .$$

It is a classical lemma that the real function $|x|$ is approximated uniformly by a sequence of polynomials $g_n(x)$ with rational coefficients. Then for each f in $\mathcal{B}$, $g_n(f)$ is in $\mathcal{B}$ and $|f|$ is approximated uniformly by $\langle g_n(f) \rangle$; thus also $|f| \in \mathcal{B}$. It follows by the definition of max, min in terms of the rational operations and absolute value that:

(7) $\quad f_1,\ldots,f_n \in \mathfrak{B} \Rightarrow \max(f_1,\ldots,f_n) \in \mathfrak{B} \wedge \min(f_1,\ldots,f_n) \in \mathfrak{B}$.

With these preliminaries out of the way, we establish (3) by a double use of countable compactness, as follows. Suppose given $f \in C(A, \mathbb{R})$. We are to find (for each $\varepsilon > 0$) a member p_n of $\mathfrak{A}$ with $\|f-p_n\| \leq \varepsilon$. First, given any x,y

(8) $\quad$ we can find n with $|f_n(x)-p_n(x)| < \varepsilon \wedge |f_n(y) - p_n(y)| < \varepsilon$.

Since f and p_n are continuous, there exists B_m such that

(9) $$y \in B_m \text{ and } [z \in B_m \Rightarrow |f(z) - p_n(z)| < \varepsilon].$$

Let $g = \{(n,m) | \forall z \in B_m |f(z) - p_n(z)| \leq \varepsilon\}$; as we know g is a set, since we need only test the condition at all q_n in B_m. Further $A \subseteq \bigcup_{(n,m)\in g} B_m$ by (9). Hence there exist a finite number $(n_1,m_1),\ldots,(n_t,m_t)$ of members of g for which $A \subseteq B_{m_1} \cup \ldots \cup B_{m_t}$. On B_{m_i} we have $|f(z) - p_{n_i}(z)| \leq \varepsilon$, so $p_{n_i}(z) \leq f(z) + \varepsilon$. Also each $p_{n_i}(x) \geq f(x) - \varepsilon$ by (8). Thus if we take $h_x = \min(p_{n_1},\ldots,p_{n_t})$ we have

(10) $\quad h_x \in \mathfrak{B}$ and $\forall z \in A(h_x(z) \leq f(z) + \varepsilon) \wedge h_x(x) \geq f(x) - \varepsilon$.

Approximating h_x by a p_k in $\mathfrak{A}$ (and replacing ε by an $\varepsilon_1 < \varepsilon/2$ throughout) we obtain:

(11) for each $x \in A$ we can find $k = k(x)$ such that

$$p_k(x) > f(x) - \varepsilon \quad \text{and} \quad \forall z \in A(p_k(z) < f(z) + \varepsilon).$$

Again by continuity there exists m with

(12) $$x \in B_m \wedge \forall z \in B_m(p_k(z) > f(z) - \varepsilon)$$

for $k = k(x)$. Now let $g' = \{(k,m) \mid \forall z \in B_m(p_k(z) \geq f(z) - \varepsilon) \wedge \forall z \in A(p_k(z) \leq f(z)+\varepsilon)\}$, which is again a set. We have $A \subseteq \bigcup_{(k,m)\in g'} B_m$ by (11) and (12), so there exist finitely $(k_1,m_1),\ldots,(k_r,m_r)$ in g' with $A \subseteq B_{m_1} \cup \ldots \cup B_{m_r}$.
Taking $h = \max(p_{k_1},\ldots,p_{k_r})$ we have

(13) $$h \in \mathfrak{B} \quad \text{and} \quad \forall z \in A\,(f(z) - \varepsilon \leq h(z) \leq f(z) + \varepsilon).$$

This procedure is carried out uniformly in ε, i.e. given ε we have found h_ε in $\mathfrak{B}$ with $\|f-h_\varepsilon\| < \varepsilon$. From that follows $f \in \overline{\mathfrak{B}}^{(s)} \equiv \mathfrak{B}$, q.e.d.

As a corollary to the theorem we obtain:

(14) <u>if</u> $\mathfrak{A}$ <u>is a countable algebra containing</u> 1 <u>and if</u> $\mathfrak{A}$ <u>separates points approximately then</u> $\overline{\mathfrak{A}}^{(s)} \equiv C(A,\mathbb{R})$, <u>so</u> $C(A,\mathbb{R})$ <u>is separable</u>.

The classical Weierstrass theorem for $C(\mathbb{R},\mathbb{R})$ is a corollary, taking $\mathfrak{A}$ to be the algebra of all polynomial functions with rational coefficients.

Remark. In the usual form of Stone-Weierstrass theorem, $\mathcal{C}$ is not assumed to be countable. To prove that one must assume full compactness of A as well as the axiom of choice.

3.8 Differentiation of real functions. We now specialize to the classical calculus of functions of one real variable, i.e. functions whose domain of definition is an interval I in $\mathbb{R}$. To define the notion of derivative, we must use the notion

$$(1) \qquad \lim_{x \to c} g(x) = L$$

for g defined on I except possibly at c. As usual this means that we have a function $\delta = \delta(\varepsilon)$ such that for each $x \in I$ with $x \neq c$, if $|x-c| < \delta$ then $|g(x) - L| < \varepsilon$. For f defined on I let $g(x) = \frac{f(x)-f(c)}{x-c}$. Then f is differentiable at c if $\lim_{x \to c} g(x)$ exists. It is easily shown that f is continuous at c in this case. f is said to be differentiable on $I_1 \subseteq I$ if we have a function f' defined on I_1 and a function $\delta = \delta(x, \varepsilon)$ such that:

$$(2) \quad x \in I_1 \wedge u \in I \wedge u \neq x \wedge |u-x| < \delta \Rightarrow \left|\frac{f(u)-f(x)}{u-x} - f'(x)\right| < \varepsilon .$$

We are primarily concerned with f which are differentiable on I except possibly at a finite (or at worst countably infinite) number of points.

The laws of derivatives are verified in the standard way. The same holds for the following main facts about differentiation, where (for simplicity) f is assumed to be defined and continuous on $[a,b]$ and differentiable on (a,b):

(3) (i) <u>if</u> f <u>has a local maximum(minimum) at</u> $x \in (a,b)$ <u>then</u> $f'(x) = 0$;

(ii) <u>there exists</u> $x \in (a,b)$ <u>with</u> $\frac{f(b)-f(a)}{b-a} = f'(x)$;

(iii) <u>if</u> $f'(x) \geq 0$ <u>on</u> (a,b) <u>then</u> $a < x_1 \leq x_2 < b \Rightarrow f(x_1) \leq f(x_2)$

(iv) <u>if</u> $f'(x) \leq 0$ <u>on</u> (a,b) <u>then</u> $a < x_1 \leq x_2 < b \Rightarrow f(x_1) \geq f(x_2)$

(v) <u>if</u> $f'(x) = 0$ <u>on</u> (a,b) <u>then</u> f <u>is constant on</u> $[a,b]$.

The statements (iii) - (v) are immediate consequences of the Mean Value Theorem (ii).

<u>Exercise</u>. Prove (3)(i),(ii).

3.9 <u>Riemann integration</u>. Suppose given a closed interval $I = [a,b]$ with $a < b$. A finite non-empty set of points $X = \{x_o, \ldots, x_n\}$ in I is presented as a finite sequence $\langle x_o, \ldots, x_n \rangle$ together with its length n. We can eliminate repetitions as follows. Let $d = \{i \leq n \mid \neg \exists j > i(x_i = x_j)\}$; d is a set and we can list it as $d = \{i_o, \ldots, i_k\}$ with $i_o < \ldots < i_k$. Then $X^o = \{x_{i_o}, \ldots, x_{i_k}\}$ has the same members as X but these are now pairwise distinct. Next, suppose given $X = \{x_o, \ldots, x_n\}$ with $x_i \neq x_j$ for

$i < j \leq n$. Let $r = \{(i,j) | x_i < x_j\}$; this is also a set, which is a linear ordering of $\{0,\ldots,n\}$. Hence we can find a permutation π such that $X^* = \{x_{\pi(0)},\ldots,x_{\pi(n)}\}$ has the same elements as X and $x_{\pi(0)} < \ldots < x_{\pi(n)}$ in the ordering of $\mathbb{R}$. By a <u>partition of</u> $[a,b]$ we mean an <u>ordered set</u> $P = \{x_0,\ldots,x_n\}$ (i.e. such that $x_0 < x_1 < \ldots < x_n$) with $x_0 = a$ and $x_n = b$. We use P, P', etc. to range over partitions throughout. P' is said to be a <u>refinement</u> of P if $P \subseteq P'$. Given any P, P' we can form a refinement of both P, P' by taking $(P \cup P')^{0*}$.

If $P = \{x_0,\ldots,x_n\}$ and $Q = \{t_0,\ldots,t_{n-1}\}$ we say that Q is meshed with P if $x_i \leq t_i \leq x_{i+1}$ for each $i < n$; we write $Q \leq P$ in this case. Given any such P,Q and $f: [a,b] \to \mathbb{R}$ form

(1) $$S_f(P,Q) = \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i).$$

When f is fixed it is dropped as a subscript. f is said to be <u>Riemann-integrable on</u> $[a,b]$ and we write $f \in \mathcal{R}[a,b]$ (or $f \in \mathcal{R}(I)$) if there is a number L and a function $\epsilon \mapsto P_\epsilon$ which associates with each (positive rational) ϵ a partition P_ϵ such that

(2) $$\forall P \subseteq P_\epsilon \; \forall Q \leq P \; \{|S(P,Q) - L| < \epsilon\}.$$

Suppose given another function $\epsilon \mapsto P'_\epsilon$ and number L' satisfying (2) for the same f. For any $\epsilon > 0$, take a refinement P of both P_ϵ and P'_ϵ and let $Q \leq P$. Then $|S(P,Q)-L| < \epsilon$ and $|S(P,Q) - L'| < \epsilon$, so

$|L - L'| < 2\epsilon$. It follows that $L = L'$. When there exists L and $\lambda\epsilon . P_\epsilon$ satisfying (2), we denote the first by

$$(3) \qquad L = \int_a^b f(x)dx = \int_a^b f .$$

From this definition is easily derived the basic linearity properties of the Riemann integral:

(4) (i) If $f, g \in \mathcal{R}[a,b]$ and $c_1, c_2 \in \mathbb{R}$ then $c_1 f + c_2 g \in \mathcal{R}[a,b]$ and

$$\int_a^b (c_1 f(x) + c_2 g(x))dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx .$$

(ii) If $a < b < c$ and $f \in \mathcal{R}[a,b]$ and $f \in \mathcal{R}[b,c]$ then $f \in \mathcal{R}[a,c]$ and

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx .$$

There is an intrinsic n.a.s.c. for f to be integrable which is due to Riemann, namely that we can find a function $\epsilon \mapsto P_\epsilon$ such that

$$(5) \qquad \forall P \supseteq P_\epsilon \; \forall Q_1 \le P_\epsilon \; \forall Q_2 \le P_\epsilon \; \{|S(P,Q_1) - S(P,Q_2)| < \epsilon\} .$$

For the proof of equivalence of this with $f \in \mathcal{R}[a,b]$, suppose first that (2) holds. Then for any $Q_1, Q_2 \le P_{\epsilon/2}$ we have $|S(P,Q_1) - S(P,Q_2)| \le |S(P,Q_1) - L| + |S(P,Q_2) - L| < \epsilon/2 + \epsilon/2 = \epsilon$, so $P_{\epsilon/2}$ serves for (8). Conversely, given $\lambda\epsilon . P_\epsilon$ satisfying (5), let $P^{(m)} = P_{(1/m)}$. We first show that if $Q_1 \leq P^{(m_1)}$ and $Q_2 \leq P^{(m_2)}$ then

(6) $|S(P^{(m_1)},Q_1) - S(P^{(m_2)},Q_2)| < \frac{1}{m}$ for $m = \max(m_1,m_2)$.

For, let $P = (P^{(m_1)} \cup P^{(m_2)})^{o*}$ be the refinement of $P^{(m_1)}$, $P^{(m_2)}$. Say $S(P^{(m_1)}, Q_1) \geq S(P^{(m_2)}, Q_2)$. We claim there are Q, $\tilde{Q} \leq P$ such that $S(P,Q) \geq S(P^{(m_1)},Q_1) \geq S(P^{(m_2)}, Q_2) \geq S(P,\tilde{Q})$. To see how these are obtained, say $P^{(m_1)} = \{x_o,x_1,\ldots,x_n\}$, and $P = \{x'_o, x'_1, \ldots, x'_k, \ldots\}$ where $\{x'_o, x'_1,\ldots,x'_k\} = P \cap [x_o,x_1]$. Also $Q_1 = \{t'_o,t'_1,\ldots,t'_{k-1},\ldots\}$ where $x'_i \leq t'_i \leq x'_{i+1}$. On $[x_o,x_1]$ choose t_o where $f(t_o) = \max_{i \leq k} f(t'_i)$. Thus $f(t_o)(x_1-x_o) \geq \sum_{i=0}^{k-1} f(t'_i)(x'_{i+1} - x'_i)$. We may proceed similarly for each of the intervals $[x_1,x_2]$, etc., to obtain $Q = \{t_o,t_1,\ldots\}$ with $S(P,Q) \geq S(P^{(m_1)}, Q_1)$. Working symmetrically with P as a refinement of $P^{(m_2)}$ and using minima with respect to the $f(t''_i)$, we determine $\tilde{Q}$ so that $S(P^{(m_2)}, Q_2) \geq S(P,\tilde{Q})$. The conclusion is that $S(P^{(m_1)}, Q_1) - S(P^{(m_2)}, Q_2) \leq S(P,Q) - S(P, \tilde{Q}) < \frac{1}{m}$, thus proving (6). Now for $P^{(m)} = \{x_o^{(m)}, \ldots, x_{n_m}^{(m)}\}$, let $Q^{(m)} = \{x_o^{(m)}, \ldots, x_{n_m-1}^{(m)}\}$, and let $s_m = S(P^{(m)}, Q^{(m)})$. It follows from (6) that $\langle s_m \rangle$ is a Cauchy sequence. Take $L = \lim_m s_m$. Then (2) is satisfied as follows: given $1/m < \epsilon$, and $P \subseteq P^{(m)}$, $Q \leq P$, consider $|s_m - S(P,Q)|$. If $s_m \geq S(P,Q)$

then as before we obtain $\tilde{Q}$ on $P^{(m)}$ such that $S(P,Q) \geq S(P^{(m)}, \tilde{Q})$. Hence $s_m - S(P,Q) \leq S(P^{(m)}, Q^{(m)}) - S(P^{(m)}, \tilde{Q}) < \frac{1}{m}$. Similarly if $s_m \leq S(P,Q)$. It follows that $|s_m - S(P,Q)| \leq \frac{1}{m} < \epsilon$.

This argument shows that when Riemann's condition (5) is verified with an explicitly given $\lambda\epsilon.P_\epsilon$ we can explicitly find $\int_a^b f$ as $\lim_m S(P^{(m)}, Q^{(m)})$. As an application of the condition we have:

(7) <u>if</u> $f \in \mathcal{R}[a,b]$ <u>and</u> $x \in [a,b]$ <u>then</u> $f \in \mathcal{R}[a,x]$ <u>and</u> $f \in \mathcal{R}[x,a]$.

For this it is sufficient to show how to satisfy (5) on both $[a,x]$ and $[x,a]$ given a function $\epsilon \mapsto P_\epsilon$ which satisfies it on $[a,b]$. We may simply take $P_\epsilon^x = (P_\epsilon \cap [a,x]) \cup \{x\}$ and $\tilde{P}_\epsilon^x = (P_\epsilon \cap [x,a]) \cup \{x\}$. The functions $\lambda\epsilon.P_\epsilon^x$, $\lambda\epsilon.\tilde{P}_\epsilon^x$ are easily used together to verify both $f \in \mathcal{R}[a,x]$ and $f \in \mathcal{R}[x,b]$. Note that from (7) and the preceding remark we can consider $\int_a^x f$ as a function $F(x)$ defined on $[a,b]$.

The main part of the Fundamental Theorem of Calculus may now be obtained.

(8) <u>If</u> F <u>is differentiable on</u> $[a,b]$ <u>and</u> $F'(x) = f(x)$ <u>for</u> $a < x < b$ <u>where</u> $f \in \mathcal{R}[a,b]$ <u>then</u> $\int_a^b f(x)dx = F(b) - F(a)$.

The proof is simply as follows. Suppose given $\lambda\epsilon.P_\epsilon$ satisfying (2) and let $\epsilon > 0$. For $P = P_\epsilon = \{x_0, \ldots, x_k\}$, write

$$F(b) - F(a) = \sum_{i=0}^{n-1} (F(x_{i+1}) - F(x_i)) = \sum_{i=0}^{n-1} F'(t_i)(x_{i+1} - x_i) ,$$

where $x_i < t_i < x_{i+1}$ is chosen by the Mean Value Theorem for derivatives (3.8(3)). Thus $F(b) - F(a) = S(P,Q)$ for a $Q \leq P$ and $|F(b) - F(a) - \int_a^b f| = |S(P,Q) - \int_a^b f| < \epsilon$. Since this is true for every ϵ we have the conclusion of (8).

Remark. In many treatments of Riemann integration for bounded functions f one associates with each P the numbers $M_i = \sup_{x_i \leq x \leq x_{i+1}} f(x)$, $m_i = \inf_{x_i \leq x \leq x_{i+1}} f(x)$ and $U_f(P) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i)$, $L_f(P) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i)$. Then f is Riemann integrable iff there exists $\lambda \epsilon . P_\epsilon$ such that $U(P_\epsilon) - L(P_\epsilon) < \epsilon$. This is simpler than the definition we have used but requires the least upper bound axiom for sets and hence cannot be done without (Proj_1) in general. It is of course applicable to integration of continuous functions without that axiom.

Exercise. Prove the linearity properties (4)(i),(ii).

3.10 Integrable functions. Suppose $D = \{d_o, \ldots, d_{m-1}\} \subseteq [a,b] = I$ and that $f : I \to \mathbb{R}$ is bounded and continuous on $I - D$. We shall show that f is Riemann integrable on $[a,b]$. For each $k > 0$ we can find $(a_i^{(k)}, b_i^{(k)})$, $0 \leq i < m_k$ such that

(1) $d_i \in (a_i^{(k)}, b_i^{(k)})$ for $i < m_k$ and $\Sigma_{i<m_k}(b_i^{(k)} - a_i^{(k)}) < \frac{1}{k}$.

Let $G^{(k)} = \bigcup_{i<m_k} b_i^{(k)}, b_i^{(k)})$ and $A^{(k)} = I - G^{(k)}$. Then we denote by $\mu(G^{(k)})$ the "length" $\Sigma_{i<m_k}(b_i^{(k)} - a_i^{(k)})$ of $G^{(k)}$.

(2) $A^{(k)}$ <u>is</u> (i) <u>seq. compact and</u> (ii) <u>countably compact</u>.

For (i) simply use that I is seq. compact and that $A^{(k)}$ is closed. For (ii), given any open cover of $A^{(k)}$, combine it with $G^{(k)}$ to get an open cover of I ; this can be reduced to a finite subcover. $A^{(k)}$ is obviously separable, with basis $\mathbb{Q} \cap A^{(k)}$. By 3.2, f is uniformly continuous on $A^{(k)}$ so we can find for each $\varepsilon > 0$ a $\delta(=\delta_k(\varepsilon))$ such that

(3) $t, t' \in A^{(k)} \wedge |t-t'| < \delta \Rightarrow |f(t) - f(t')| < \frac{\varepsilon}{2(b-a)}$.

Further, given any subinterval $[u,w]$ of I we can compute

$$\sup_{u \leq x \leq \omega} f(x) \text{ and } \inf_{u \leq x \leq \omega} f(x) \text{ and thence } \sup_{u \leq t,t' \leq \omega} |f(t)-f(t')| .$$

Let $\ell = \sup_{a \leq t,t' \leq b} |f(t)-f(t')| + 1$. Now, given any $\varepsilon > 0$, choose k such that $\mu(G^{(k)}) < \frac{\varepsilon}{2\ell}$. Then, for that k , choose δ to satisfy (3). Let P_ε be a partition which includes D as a subset and is such that each interval in P_ϵ has length $< \delta$. Consider any $P \supseteq P_\epsilon$, $P = \{x_0, \ldots, x_n\}$. For each subinterval of P_ϵ we have either $[x_{i+1}, x_i] \subseteq A^{(k)}$ or (x_{i+1}, x_i) disjoint from $A^{(k)}$. Given any $Q, Q' \leq P$,

$$|S(P,Q)-S(P,Q')| \leq \Sigma_i^{(1)} |f(t_i)-f(t_i')|(x_{i+1}-x_i) + \Sigma_i^{(2)} |f(t_i)-f(t_i')|(x_{i+1}-x_i),$$

where the first sum is extended over those intervals in P contained in $A^{(k)}$, and the second over the remaining intervals. Now

$$\Sigma_i^{(1)} |f(t_i) - f(t_i')|(x_{i+1}-x_i) < \frac{\epsilon}{2(b-a)} \Sigma_i^{(1)} (x_{i+1}-x_i) \leq \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2} .$$

On the second sum we have

$$\Sigma_i^{(2)} |f(t_i) - f(t_i')|(x_{i+1}-x_i) < \ell \cdot \Sigma_i^{(2)} (x_{i+1}-x_i) < \ell \cdot \frac{\epsilon}{2\ell} = \frac{\epsilon}{2} .$$

Hence $|S(P,Q) - S(P,Q')| < \epsilon$. With this choice of P_ϵ we have thus satisfied Riemann's condition of integrability, so that we have proved:

(3) __if__ $f : [a,b] \to \mathbb{R}$ __and__ f __is bounded and continuous except possibly at finitely many points, then__ $f \in \mathcal{R}[a,b]$.

We also have

(4) <u>under the same hypotheses as</u> (3),

(i) $\int_a^b f = p(b-a)$ <u>for some</u> p <u>with</u> $m = \inf_{a \le x \le b} f(x) \le p \le \sup_{a \le x \le b} f(x) = M$, <u>and</u>

(ii) <u>for</u> $F(x) = \int_a^x f$ <u>we have</u> $F'(x) = f(x)$ <u>for each</u> $x \in [a,b]$.

(i) is the Mean Value Theorem for integration; it is immediate from $m(b-a) \le S(P,Q) \le M(b-a)$ for each P and $Q \le P$. To prove (ii), consider any $a \le x < u \le b$. Then $F(u) - F(x) = \int_x^u f = p(u-x)$ for some p between $m[x,u] = \inf_{x \le t \le a} f(t)$ and $M[x,u] = \sup_{x \le t \le u} f(t)$. Hence $\lim_{u \to x} \frac{F(u)-F(x)}{u-x} = f(x)$ and we can provide the required rate of convergence to verify this.

<u>Remark</u>. The functions of classical interest satisfy (3). However, it is possible to describe a much wider class of Riemann integrable functions in terms of the theory of measure, namely (*) those which are <u>continuous outside a set of measure</u> 0 (and, in fact only those). The measure of an open set G is defined to be $\mu(G) = \Sigma_{i=0}^{\infty}(b_i - a_i)$ where $G \equiv \cup_i(a_i, b_i)$ is the disjoint representation of G. A set D is said to be of measure 0 if we have a decreasing sequence $\langle G^{(k)} \rangle$ containing D such that for each $\epsilon > 0$ we can find k with $\mu(G^{(k)}) < \epsilon$. Every countable set is of measure 0. The germs of the result (*) are already contained in the above proof. We delay further discussion to §4.

3.11 <u>Picard's existence theorem for ordinary differential equations.</u>

Let A, B be closed and bounded intervals in $\mathbb{R}$, and $D = A \times B$. Suppose $f : D \to \mathbb{R}$ is continuous; then we can find K with

(1) $$|f(x,y)| \leq K$$

for $(x,y) \in D$. For the theorem to be proved it is also assumed that f satisfies a <u>Lipschitz condition</u>

(2) $$|f(x,y_1) - f(x,y_2)| \leq M \cdot |y_1 - y_2|$$

for some M and all (x,y_1), $(x,y_2) \in D$. (This is satisfied, for example, if $\frac{\partial f}{\partial y}$ is continuous in D.) Given $(x_o, y_o) \in D$, to solve the differential equation with the initial value condition

(3) (i) $g'(x) = f(x,g(x))$

(ii) $g(x_o) = y_o$

it is necessary and sufficient that g solves the integral equation

(4) $$g(x) = y_o + \int_{x_o}^{x} f(t,g(t))dt.$$

The Picard theorem is that, under the given hypotheses (of continuity of f and (2)) if y_o is interior to B then

(5) <u>we can find</u> ε <u>and</u> g <u>for which</u> g <u>uniquely satisfies</u> (3) <u>when</u> $x \in [x_o - \varepsilon, x_o + \varepsilon] \cap A$.

For the proof, take $b > 0$ with $S(y_o ; b) \subseteq B$. Let

(6) $$\epsilon < \min(\frac{b}{K}, \frac{1}{M}) \quad \text{and} \quad I = [x_o - \epsilon, x_o + \epsilon] \cap A.$$

Consider the space

(7) $$X = \{g \in C(I, \mathbb{R}) | \forall x \in I(|g(x) - y_o| \leq b)\}.$$

This is a closed subspace of $C(I, \mathbb{R})$ and is complete and separable by 3.6 and 3.7. Define the mapping T on X by

(8) $$T(g) = y_o + \int_{x_o}^{x} f(t, g(t))dt.$$

We then have

(9) $$T : X \to X$$

since $|T(g) - y_o| \leq |\int_{x_o}^{x} f(t,g(t))dt| \leq K \cdot |x - x_o| \leq K \cdot \epsilon \leq b$. Also

(10) T <u>is a contraction mapping</u>,

for $|T(g_1) - T(g_2)| \leq |\int_{x_o}^{x} f(t, g_1(t) - f(t, g_2(t_2))dt| \leq M \cdot |x - x_o| < 1$.

By the contraction mapping theorem of 2.14,

(11) $T(g) = g$ <u>has a unique solution</u> g <u>in</u> X.

Remarks:

(i) ε can be chosen independent of b, K if (2) is satisfied on $A \times \mathbb{R}$. Then by a finite number of shifts to the right and left we can further obtain a solution on all of A.

(ii) There are simple generalizations of the theorem to simultaneous equations $g_i'(x) = f_i(x, g_o(x), \ldots, g_n(x))$ $(0 \leq i \leq n)$ and thence to higher order equations $g^{(n)}(x) = f(x, g(x), \ldots, g^{(n-1)}(x))$.

3.12 *Series of functions*. In this and the remaining sections of § 3 we set down various results without proof which can be drawn as consequences in VFT↾ + (μ) of the preceding general material in a standard way. For details of proofs see such references as Rudin ~~1953~~ 1976 or Hoffman 1975.

We work here in $C(A, \mathbb{R})$ where A is a closed and bounded interval $A = [a,b]$. Uniformly convergent sequences (and hence series) of functions have the following good properties:

(1) *If* $\langle f_n \rangle$ *converges uniformly on* $[a,b]$ *to* f *then*

$$\int_a^b f(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx.$$

(2) *If the sequence of derivatives* $\langle f_n' \rangle$ *is defined on* $[a,b]$ *and converges uniformly to* g *and if* $\langle f_n \rangle$ *converges to* f *then* $f' = g$.

The *Weierstrass M-test* is a simple sufficient test for uniform convergence:

(3) <u>if</u> $|f_n(x)| \leq M_n$ <u>for all</u> n <u>and</u> $x \in [a,b]$ <u>and if</u> $\Sigma_{n=0}^{\infty} M_n$ <u>converges then</u> $\Sigma_{n=0}^{\infty} f_n$ <u>converges uniformly on</u> $[a,b]$.

This is applied in particular to power series by comparison with the geometric series $\Sigma_{n=0}^{\infty} r^n$ which converges to $\frac{1}{1-r}$ when $|r| < 1$ and diverges when $|r| \geq 1$. Given a sequence $\langle a_n \rangle$, the formal series $\Sigma_{n=0}^{\infty} a_n(x-x_o)^n$ is called a <u>power series about</u> x_o.

(4) <u>If</u> $r = \lim_{n \to \infty} |\frac{a_n}{a_{n+1}}|$ <u>exists or is</u> $+\infty$ <u>then</u> $\Sigma_{n=0}^{\infty} a_n(x-x_o)^n$ <u>converges for each</u> x <u>with</u> $|x-x_o| < r$ <u>and diverges for</u> $|x-x_o| > r$. <u>Furthermore, for</u> $f_n(x) = a_n(x-x_o)^n$, <u>the convergence of</u> $\Sigma_{n=0}^{\infty} f_n$ <u>is uniform on</u> $[x_o-h, x_o+h]$ <u>for each</u> $h < r$.

Such r is called the <u>radius of convergence</u> of the power series. A better result is usually formulated with $r = \frac{1}{s}$ where $s = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$. However, (4) serves our present purposes.) The series obtained by formally integrating $\Sigma_{n=0}^{\infty} a_n(x-x_o)^n$ term by term is $\Sigma_{n=0}^{\infty} a_n^*(x-x_o)^n$ where $a_o^* = 0$, $a_{n+1}^* = \frac{a_n}{n+1}$; that obtained by formally differentiating term by term is $\Sigma_{n=0}^{\infty} a_n'(x-x_o)^n$ where $a_n' = n\,a_{n+1}$. If $r = \lim_{n \to \infty} |\frac{a_n}{a_{n+1}}|$ exists or is $+\infty$ then also $\lim_{n \to \infty} |\frac{a_n^*}{a_{n+1}^*}| = r = \lim_{n \to \infty} |\frac{a_n'}{a_{n+1}'}|$. Thus we conclude that the series $f(x) = \Sigma_{n=0}^{\infty} a_n(x-x_o)^n = \Sigma_{n=0}^{\infty} f_n(x)$ is uniformly convergent on each interval $[x_o - h, x_o + h]$ with $h < r$ and $\int_a^b f(x)dx = \Sigma_{n=0}^{\infty} \int_a^b f_n(x)dx$ for

$x_o - h \leq a \leq b \leq x_o + h$, and that further $f'(x) = \Sigma_{n=0}^{\infty} f_n'(x)$ for each $x \epsilon [x_o - h, x_o + h]$. By successive differentiation we thus obtain

(5) $$a_n = \frac{f^{(n)}(x_o)}{n!} \quad \text{and} \quad f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!} (x-x_o)^n \quad \text{in} \quad (x_o - r, x_o + r).$$

3.13 <u>The classical transcendental functions.</u> Using the expected laws of differentiation for the functions $E(x) = e^x$, $C(x) = \cos x$, $S(x) = \sin x$, we may define these by the power series

(1) (i) $$E(x) = \Sigma_{n=0}^{\infty} \frac{1}{n!} x^n$$

(ii) $$C(x) = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

(iii) $$S(x) = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad .$$

For each of these the radius of convergence is found to be $r = +\infty$, so the series converge uniformly and are integrable and differentiable term-by-term on each bounded interval. Then one can deduce the familiar and characteristic facts concerning these functions such as:

(2) (i) $E'(x) = E(x)$

(ii) $E(x)$ is strictly increasing

(iii) $E(x+y) = E(x) \cdot E(y)$

(iv) $E(x) = e^x$ where $e = E(1)$.

(v) $E(x) > 0$.

For (ii), one needs a result about multiplication of absolutely convergent series, so as to write

$$\Sigma_{n=0}^{\infty} \frac{x^n}{n!} \; \Sigma_{m=0}^{\infty} \frac{y^m}{m!} = \Sigma_{n=0}^{\infty} \Sigma_{k=0}^{n} \frac{x^k y^{n-k}}{k!(n-k)!} = \Sigma_{n=0}^{\infty} \frac{1}{n!} \Sigma_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \Sigma_{n=0}^{\infty} \frac{1}{n!}(x+y)^n .$$

For (iv) it is first shown that $E(nx) = (E(x))^n$ and $E(\frac{x}{n}) = (E(x))^{1/n}$ so $E(q) = e^q$ for each rational q. Then $E(x) = e^x$ by continuity. Since $E(x)\,E(-x) = E(0) = 1$, we have $E(-x) = 1/E(x)$; but for $x > 0$, $E(x) > 0$ so also $E(-x) > 0$.

Similarly we obtain

(3) (i) $C'(x) = - S(x)$, $S'(x) = C(x)$

(ii) $C(-x) = C(x)$, $S(-x) = - S(x)$

(iii) $C(0) = 1$, $S(0) = 0$

(iv) $C(x+y) = C(x)\,C(y) - S(x)\,S(y)$

$S(x+y) = S(x)\,C(y) + C(x)\,S(y)$

(v) $C^2(x) + S^2(x) = 1$.

It may be shown that there is a least positive number x_1 with $C(x_1) = 0$; denoting this by $\pi/2$ we get

(4) (i) $C(\pi/2) = 1$, $S(\pi/2) = 0$

(ii) $C(\pi) = -1$, $S(\pi) = 0$

(iii) $C(2\pi) = 1$, $S(2\pi) = 0$

Finally, these functions are periodic of period 2π ,

(5) $$C(X+2\pi) = C(x) , \quad S(x+2\pi) = S(x).$$

The usual techniques of integration of trigonometric functions can be applied to give for any integers n, m:

(6) (i) $$\int_0^{2\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

(ii) $$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

(iii) $$\int_0^{2\pi} \cos(nx)\sin(mx)dx = 0, \quad \text{all } n, m .$$

3.14 <u>Fourier series.</u> Assume throughout this section that

(1) (i) f <u>is piecewise continuous on</u> $[0, 2\pi]$ <u>and</u>

(ii) f <u>is periodic of period</u> 2π .

The hypothesis (i) means that $f(x^+) = \lim_{h \to 0^+} f(x+h)$ and $f(x^-) = \lim_{h \to 0^+} f(x-h)$ exist for each x and that $f(x^+) \neq f(x^-)$ for at must a finite number of x in $[0,2\pi]$. If $f(x^+) = f(x^-)$ there is at most a "removable" discontinuity of f at x, otherwise a "jump" discontinuity. In seeking an expansion of the form

(2) $$f(x) = c_o + \Sigma_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

one is led to multiplying by $\cos(mx)$ or $\sin(mx)$ and integrating term by term. Formally applying 3.13(6) brings us to take:

(3) (i) $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos(nx)\,dx \qquad (n = 0, 1, 2, \dots)$$

(ii) $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin(nx)\,dx \qquad (n = 1, 2, \dots)$$

and

(iii) $$c_o = \frac{a_o}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx .$$

The series on the r.h.s. of (2) with coefficients defined by (3) is called the <u>Fourier series</u> of f. There are a vast number of results about convergence of Fourier series, both in the usual and in other senses. The easiest to obtain are the following:

(4) (i) <u>If</u> f <u>is differentiable at</u> x <u>then the Fourier series of</u> f <u>converges to</u> f <u>at</u> x.

(ii) <u>If</u> f <u>is piecewise differentiable then the Fourier series converges to</u> $f(x^+) + f(x^-)$ <u>for each</u> x.

We also have Fejer's theorem:

(5) $$\lim_{n\to\infty} \sigma_n(x) = \frac{f(x^+)+f(x^-)}{2} \quad \underline{\text{where}} \quad \sigma_n(x) = \frac{1}{n}\Sigma_{j=0}^{n} s_j(x)$$

<u>and</u> $$s_n(x) = \frac{a_o}{2} + \Sigma_{k=0}^{n} (a_k \cos kx + b_k \sin k_x).$$

These results can be used to obtain existence and uniqueness theorems for familiar partial differential equations given by physical problems. For a basic treatment cf. Berg and McGregor 1966.

3.15 Complex analysis. The fundamental notions of analysis for functions $f : \mathbb{C} \to \mathbb{C}$ are now definable and treatable in any one of the standard ways. These comprise the notion of differentiable function and the operation of differentiation, integration along a (piecewise smooth) path, analytic functions, independence of path for integration of analytic functions, Cauchy's integral formulas, and expansion of analytic functions in power series. The results for the latter are just as for real power series 3.12. Then the transcendental functions e^z, cos z, sin z can be defined by power series for $z \in \mathbb{C}$ and the properties of 3.13 extended to complex numbers. Examples of further results which are easily established in $VFT\upharpoonright +(\mu)$ are the maximum modulus theorem, Liouville's theorem and the Fundamental Theorem of Algebra. In fact one may expect from Bishop's 1967 Constructive treatment of Complex analysis which reaches the Riemann mapping theorem that this development can be pushed somewhat further. The conclusion of §3 is thus that the body if not all of classical analysis through the 19th century can be carried out in the restricted theoretical framework $VFT\upharpoonright +(\mu)$.

4. *Developments in* $\mathrm{VFT}\restriction + (\mu)$: *modern analysis*. In this section we describe how two basic parts of modern analysis are to be dealt with in $\mathrm{VFT}\restriction + (\mu)$. The first is *Lebesgue's theory of measure and integration*, here dealt with only for the concrete spaces $\mathbb{R}^k$. It is frequent in modern treatments to start with the notion of *outer measure* $\mu^*(X)$, from which the notion of measurable set is derived. This cannot be done in our restricted theory because the definition of $\mu^*(X)$ requires the inf operation on sets of reals in an essential way. It is however possible to define measurable sets X and their measure $\mu(X)$ directly. Instead of taking that route to the theory of integration we follow Riesz' approach, which is nicely presented in Shilov and Gurevich 1965. This uses only the concept of set of measure 0 to proceed directly to the Lebesgue integral. Every integrable function is represented as a difference of two monotone sequences of step functions; it is a second essential point of our development that we operate only with such *presentations* of integrable functions. The theory of measurable sets is obtained as a corollary to integration theory.

The latter half of this section takes up *functional analysis*, i.e. the theory of linear operators on normed linear spaces, Banach spaces and Hilbert spaces. The main examples of interest are various spaces of functions, most of which are separable; the hypothesis of separability is assumed throughout. We concentrate on fundamental theorems about linear functionals and operators which require *prima-facie* impredicative arguments: the Riesz representation theorem, Hahn-Banach theorem, uniform boundedness theorem and the open mapping theorem. By relying heavily on separability we are able to see these through in our setting as well. The section concludes with principal results of the spectral theory of compact self-adjoint operators on a Hilbert space. A good reference for the part of functional analysis treated is Kreyszig 1978.

4.1 <u>Sets of measure 0 and preliminaries to Lebesgue integration theory.</u> For simplicity we shall deal with the theory on a finite bounded interval $I = [a,b]$ in $\mathbb{R}$ of length $|I| = (b-a)$. Later it will be indicated how this is to be extended to unbounded intervals as well as any $\mathbb{R}^p$.

The usual approach begins with the theory of measure which in turn begins with the notion of <u>outer measure</u> μ^*, defined by $\mu^*(X) = \inf\{|G| \; |X \subseteq G\}$ where G ranges over open sets and $|G|$ is the length of G, i.e. $|G| = \sum_n |I_n|$ when $G \equiv \bigcup_n I_n$ is represented as a union of pairwise disjoint intervals. However, $\mu^*(X)$ cannot in general be proved to exist in VFT + (μ). It is possible, though, to explain the concept of <u>measurable set</u> X and to define the measure $\mu(X)$ without passing through μ^*. Namely, X is said to be measurable if we can find a pair $\langle G_n \rangle$, $\langle G'_n \rangle$ of sequences of open sets such that for each n, $X \subseteq G_n$, $[a,b] - X \subseteq G'_n$ and for each $\epsilon > 0$ we can find n with $|G_n| + |G'_n| \leq (b-a) + \epsilon$. Then $\mu(X)$ is taken to be $\inf_n |G_n|$. When working with infinite operations on measurable sets it is essential to operate on the associated pairs $(\langle G_n \rangle, \langle G'_n \rangle)$, which may be called <u>measure-covers for</u> X. It is easily seen that if X is measurable then so also is $[a,b] - X$ and $\mu([a,b] - X) = (b-a) - \mu(X)$. Further, every subinterval J of $[a,b]$ is measurable, with

$\mu(J) = |J|$. The main result is <u>countable-additivity</u>. If $\langle X_m \rangle$ is a sequence of pairwise disjoint measurable sets with sequence of measure-covers $(\langle G_{m,n} \rangle_n, \langle G'_{m,n} \rangle_n)$ for each m then $\bigcup_m X_m$ is measurable and $\mu(\bigcup_m X_m) = \Sigma_m \mu(X_m)$. The proof of this requires a number of steps, which may be followed in Goldberg 1976, § 11.1 - 11.3. (The use of μ^* there is inessential, unlike some other presentations of the subject, in particular those which use Carathéodory's definition). One consequence of the theory is that every open set G is measurable and $\mu(G) = |G|$. All of this can be carried out in $VFT\upharpoonright + (\mu)$.

Even after the theory of measurability is developed there is still a certain amount of work to be done in order to set up the theory of the integral. Instead we take here an approach due to Riesz which leads directly to the integral, using only the notion of <u>set of measure</u> 0 at the outset. This is presented in Shilov 1965 Ch.IV, which we follow very closely. To start with, X is said to be of measure 0 if we have a sequence $\langle G_n \rangle$ of open sets such that

(1) (i) $X \subseteq G_n$ <u>for each</u> n, <u>and</u>

(ii) <u>given</u> $\epsilon > 0$ <u>we can find</u> n <u>with</u> $|G_n| < \epsilon$.

When operating on sets of measure 0 we assume given an associated sequence $\langle G_n \rangle$ satisfying (1). If X is of measure 0 we write $\mu^*(X) = 0$ (this does not give meaning to μ^* more generally). The basic result here is that

(2) <u>if</u> $\langle X_m \rangle$ <u>is a sequence of sets of measure</u> 0 <u>then so also is</u> $\bigcup_m X_m$.

Implicit in the hypothesis is that we are given a double sequence $\langle G_{m,n} \rangle$ such that for each $m, n, X_m \subseteq G_{m,n}$ and such that $|G_{m,n}|$ can be made as small as we please by taking n sufficiently large. Thus, given m and $\epsilon > 0$, choose n_m with $|G_{m,n_m}| < \frac{\epsilon}{2^{m+1}}$, then $\bigcup_m X_m$ is covered by $\bigcup_m G_{m,n_m}$ and $|\bigcup_m G_{m,n_m}| \leq \sum_m |G_{m,n_m}| < \epsilon$. This proves (2).

It is trivial that each point has measure 0, so by (2) every countable set has measure 0. The Cantor set is an example of an uncountable set of measure 0.

X is said to be of <u>full measure</u> if $[a,b] - X$ is of measure 0. A property $P(x)$ of points of $[a,b]$ is said to hold <u>almost everywhere</u> <u>(a.e.)</u> if $\{x|P(x)\}$ includes a set of full measure.

By a <u>step-function</u> h we mean one which is piecewise continuous and which is constant between successive points of discontinuity, i.e. for which we have a partition $a = x_0 < x_1 < \ldots < x_k = b$ with h constant on (x_{i-1}, x_i). A function f is said to be <u>measurable</u> if we have a sequence of step functions $\langle h_n \rangle$ with

(3) $$f(x) = \lim_n h_n(x) \quad \text{(a.e.)},$$

that is $\langle h_n(x) \rangle$ converges to $f(x)$ almost everywhere. It is easily seen that the class S of step functions is closed under linear combination, multiplication, division (by non-zero denominators) and absolute value. By passage to limits the same holds for measurable functions. Further for two step functions h_1, h_2 we also have that $\max(h_1,h_2)$ and $\min(h_1,h_2)$ are step functions. The measurable functions are consequently closed under max and min. Then it follows that if f is measurable so also are $f^+ = \max(f,0)$ and $f^- = \max(0,-f)$. The relations $f = f^+ - f^-$ and $|f| = f^+ + f^-$ are useful in further work.

Given a sequence $\langle f_n \rangle$ we write $f_n \searrow$ if the sequence is <u>monotone decreasing</u>, i.e. $f_0(x) \geq f_1(x) \geq \ldots$ for all $x \in [a,b]$. We write $f_n \searrow f$ if also $\lim_n f_n(x) = f(x)$ (a.e.). Similarly for $f_n \nearrow$ and $f_n \nearrow f$. Let h be a step-function; we put $\int h = \int_a^b h(x)dx$, which is defined by Riemann integration or simply as $\Sigma_{i=1}^k h(x_i')(x_i - x_{i-1})$ with $x_i' \in (x_{i-1}, x_i)$ and h constant on (x_{i-1}, x_i). This is linear in h and also satisfies the following order and limit properties:

(4) (i) $h_1 \geq h_2$ <u>implies</u> $\int h_1 \geq \int h_2$;

(ii) $h \geq 0$ <u>implies</u> $\int h \geq 0$;

(iii) $h_n \searrow 0$ <u>implies</u> $\lim_n \int h_n = 0$;

(iv) <u>if</u> $h_n \geq 0$, $h_n \searrow$ <u>and</u> $\lim_n \int h_n = 0$ <u>then</u> $h_n \searrow 0$.

Both (i) and (ii) are trivial. For the proof of (iii), let X_n be the set of points of discontinuity of h_n and $X = \bigcup_n X_n$; let Y be $\{x \mid \lim_n h_n(x) \neq 0\}$, and $Z = X \cup Y$. Since X is countable and $\mu^*(Y) = 0$ also $\mu^*(Z) = 0$. Let $G = \bigcup_m I_m$ be an open cover of Z with $|G| < \epsilon$ (where I_m are disjoint open intervals). When $x \notin Z$ we can find n such that $h_n(x) < \epsilon$. Thus if we take $\langle I_k' \rangle$ to be an enumeration of those open intervals on each of which some h_n is constant and $< \epsilon$, we have $[a,b] \subseteq \bigcup_m I_m \cup \bigcup_k I_k'$. By compactness, we can find a finite subcover $[a,b] \subseteq \bigcup_{m=1}^p I_m \cup \bigcup_{k=1}^q I_k'$. If I_k' is an interval of constancy of h_{n_k} and $r = \max_{1 \le k \le q} n_k$ then by monotonicity, $h_r < \epsilon$ on $\bigcup_{k=1} I_k'$. On the other hand $h_r \le h_1$ and hence h_r is bounded by a certain constant M on $\bigcup_m I_m$. Thus

$$\int h_r \le M \cdot \epsilon + \epsilon(b-a).$$

It follows that $\lim_n \int h_n = 0$.

For the proof of (iv), under the hypothesis, $g(x) = \lim_n h_n(x)$ exists for each x. It is to be shown that $g(x) = 0$ (a.e.). Let $X_m = \{x | g(x) \ge \frac{1}{m}\}$ for $m = 1, 2, \ldots$ so that $\bigcup_m X_m = \{x | g(x) > 0\}$. It is thus sufficient to show that each X_m is of measure 0. Let Y_m be the set of points in X_m at which all h_n are continuous, so X_m is Y_m plus a countable set. It is thus sufficient to show that $\mu^*(Y_m) = 0$. Each $h_n \ge \frac{1}{m}$ on Y_m. Let $d_n = \sum_k |I_{n,k}|$ where h_n is constant $\ge 1/m$ on $I_{n,k}$. Then $\int h_n \ge d_n \cdot \frac{1}{m}$ and hence $d_n \le m \int h_n$, so $\lim_m d_m = 0$. But $\langle I_{n,k} \rangle_k$ is an open cover of Y_m so $\mu^*(Y_m) = 0$.

4.2 Monotone limits of step functions. Define S^+ to be the class of functions for which we can find a monotone increasing sequence of step functions approaching f a.e. and whose integrals are uniformly bounded, i.e. for some C

(1) (i) $h_n \nearrow f$ and

(ii) $\int h_n \leq C$ for all n.

Obviously each f in S^+ is measurable. For f in S^+ we define

(2) $$\int f = \lim_n \int h_n .$$

It is to be shown that $\int f$ is independent of the choice of sequence satisfying (1). More generally, if $\langle h_n \rangle$, $\langle k_n \rangle$ are sequences of step functions with $\int h_n \leq C$, $\int k_n \leq C$ then

(3) $h_n \nearrow f$, $k_n \nearrow g$ <u>and</u> $f(x) \leq g(x)$ (a.e.) <u>implies</u> $\lim_n \int h_n \leq \lim_n \int k_n$.

To prove this, fix any m and form the monotonic decreasing sequence of step functions $h'_n = h_m - k_n$. This has limit $h_m - g$ which is $\leq (f-g) \leq 0$ (a.e.). Taking the positive parts $(h'_n)^+$ we get $(h_m - k_n)^+ \searrow 0$ so $\lim_n \int (h_m - k_n)^+ = 0$ by 4.1 (4) (iii). Now $\int (h_m - k_n)^+ \geq \int (h_m - k_n) = \int h_m - \int k_n$. Taking the limit over n we conclude that $0 \geq \int h_m - \lim_n k_n$, i.e. $\int h_m \leq \lim_n \int k_n$. Hence $\lim_m \int h_m \leq \lim_n \int k_n$ and (3) is proved.

It is easily proved that

(4) <u>if</u> f <u>is continuous on</u> $[a,b]$ <u>then</u> $f \in S^+$ <u>and</u> $\int f$ <u>is the same as the Riemann integral</u> $\int_a^b f(x)dx$.

We simply use the lower sums for f over any partition to give the approximating step functions.

The following are directly verified as consequences of the corresponding properties for step functions.

(5) <u>Suppose</u> $f,g \in S^+$, <u>and</u> $c \geq 0$. <u>Then</u> $(f+g)$, (cf), $\underline{\max}(f,g)$, $\underline{\min}(f,g)$ <u>and</u> $f^+ = \underline{\max}(f,0)$ <u>all belong to</u> S^+; <u>furthermore</u> $\int(f+g) = \int f + \int g$ <u>and</u> $\int cf = c\int f$.

A sequence $\langle f_m \rangle$ of elements of S^+ is regarded as given together with a double sequence $\langle h_{mn} \rangle$ of step functions and a sequence of bounds $\langle C_m \rangle$ such that for each m, $h_{mn} \nearrow f_m$ and $\int h_{mn} \leq C_m$. The following shows that the property (1) defining S^+ need not be iterated.

(6) <u>If</u> $f_m \in S^+$ <u>for each</u> m <u>and</u> $f_m \nearrow f$ <u>and</u> $\int f_m \leq C$ <u>then</u> $f \in S^+$ <u>and</u> $\int f = \lim_m \int f_m$.

The proof uses $h_m = \max(h_{1m}, \ldots, h_{mm})$. This forms a monotone increasing sequence with $h_m \leq \max(f_1, \ldots, f_m) = f_m$, hence $\int h_m \leq \int f_m \leq C$ by (2), (3). Then if we take $f^* = \lim_m h_m$ we have $f^* \in S^+$ and $\int f^* = \lim_m \int h_m$.

From $h_{km} \leq h_m \leq f_m$ for $k \leq m$ we get in the limit $f_k \leq f^* \leq f$ so that $f^* = f$ (a.e.). Using the same inequalities, it also follows that $\int f = \int f^* = \lim_m \int f_m$.

As a corollary to (6) we have:

(7) *if* $g_n \in S^+$ *for each* n *and* $g_n \geq 0$ *and the integrals of the partial sums* $\int \sum_{k=0}^{n} g_k$ *are uniformly bounded then* $f = \sum_{k=0}^{\infty} g_k$ *is in* S^+ *and* $\int f = \sum_{k=0}^{\infty} \int g_k$.

4.3 *The Lebesgue integral*. A function φ is called *Lebesgue integrable* if

(1) $$\varphi = (f-g) \quad \textit{for} \quad f,g \in S^+ .$$

When this holds we put $\varphi \in L$. Clearly $S^+ \subseteq L$. The following is easily established.

(2) (i) *If* $\varphi_1, \varphi_2 \in L$ *then* $(\varphi_1 + \varphi_2) \in L$;

(ii) *if* $\varphi \in L$ *and* $c \in \mathbb{R}$ *then* $c\varphi \in L$.

We also obtain

(3) $$\varphi \in L \quad \textit{implies} \quad |\varphi| , \ \varphi^+ \ \textit{and} \ \varphi^- \ \textit{are in} \ L .$$

For if $\varphi = (f-g)$ then $|\varphi| = \max(f,g) - \min(f,g)$; then by solving for φ^+, φ^- from $\varphi = \varphi^+ - \varphi^-$, $|\varphi| = \varphi^+ = \varphi^-$ we get also $\varphi^+, \varphi^- \in L$. Finally,

(4) $\varphi_1, \varphi_2 \in L$ <u>implies</u> <u>max</u>(φ_1, φ_2) <u>and</u> <u>min</u>$(\varphi_1, \varphi_2) \in L$

by $\max(\varphi_1, \varphi_2) = (\varphi_1 + \varphi_2)^+ - \varphi_2$, $\min(\varphi_1, \varphi_2) = -\max(-\varphi_1, -\varphi_2)$. The integral is extended to L by

(5) $\int \varphi = \int f - \int g$ <u>when</u> $\varphi = (f-g)$ <u>with</u> $f, g \in S^+$.

This is independent of the representation (1) of φ since if $f-g = f_1 - g_1$ we have $f + g_1 = g + f_1$ and so by 4.2(5), $\int f + \int g_1 = \int g + \int f_1$. In particular operation $\int(-)$ on L is an extension of that on S^+ .

Now we can derive the properties of L directly from corresponding properties for S^+ .

(6) (i) $\int(-)$ <u>is linear on</u> L ;

(ii) <u>if</u> $\varphi \in L$ <u>and</u> $\varphi \geq 0$ <u>then</u> $\int \varphi \geq 0$; <u>and</u>

(iii) <u>if</u> $\varphi_1, \varphi_2 \in L$ <u>and</u> $\varphi_1 \geq \varphi_2$ <u>then</u> $\int \varphi_1 \geq \int \varphi_2$.

The following lemma is convenient for our further work.

(7) <u>Given</u> $\varphi \in L$ <u>and</u> $\epsilon > 0$ <u>we can find</u> $f, g \in S^+$ <u>with</u> $\varphi = f-g$, <u>and</u> $g \geq 0$ <u>and</u> $\int g < \epsilon$; <u>further if</u> $\varphi \geq 0$ <u>then</u> $f \geq 0$.

To do this, suppose given $\varphi = (f_o - g_o)$ with $f_o, g_o \in S^+$; take a sequence of step functions $h_n \nearrow g_o$ with $\int g_o = \lim_n \int h_n$, and choose m so large that $\int g_o - \int h_m < \epsilon$. Then we re-represent φ by

$$\varphi = (f_o - h_m) - (g_o - h_m).$$

Now $g_o - h_m$ is in S^+ because it is the limit of the increasing sequence $(h_n - h_m)$; similarly for f. If $\varphi \geq 0$ then $(f-h_m) \geq (g-h_m) \geq 0$.

When dealing with sequences $\langle \varphi_n \rangle$ or series $\Sigma_n \varphi_n$ of elements of L, it is assumed that we are presented with a sequence of pairs $\langle f_n \rangle$, $\langle g_n \rangle$ such that $\varphi_n = f_n - g_n$ and $f_n, g_n \in S^+$; and then, further, that each f_n, g_n has an associated sequence of step functions as explained for S^+. The first main result is the following theorem of B. Levi, which extends 4.2(6):

(8) <u>Suppose $\varphi_n \in L$ and $\varphi_n \geq 0$ for each n and that the integrals of the partial sums $\int \sum_{k=1}^{n} \varphi_k$ are uniformly bounded; then $\varphi = \sum_{k=1}^{\infty} \varphi_k \in L$ and $\int \varphi = \Sigma_k \int \varphi_k$.</u>

For the proof, use (7) to write each φ_k as $\varphi_k = f_k - g_k$, where $f_k, g_k \in S^+$, $f_k \geq 0$, $g_k \geq 0$ and $\int g_k < \frac{1}{2^{k+1}}$. We thus have $\int \sum_{k=0}^{n} g_k < 1$. By 4.2(6), the function $g = \sum_{k=0}^{\infty} g_k$ belongs to S^+ and $\int g = \Sigma_k \int g_k$. Furthermore the same conclusion holds for $f = \sum_{k=0}^{\infty} f_k$ since each $f_k \geq 0$ and

$$\int \sum_{k=0}^{n} f_k = \int \sum_{k=0}^{n} \varphi_k + \int \sum_{k=0}^{n} g_k \leq C + 1 .$$

The result (8) follows directly.

The monotone convergence theorem for Lebesgue integration is now a corollary:

(9) if each $\psi_n \in L$ and $\psi_n \nearrow \psi$ and $\int \psi_n$ is uniformly bounded then $\psi \in L$ and $\int \psi = \lim_n \int \psi_n$.

For the proof simply take $\varphi_n = \psi_{n+1} - \psi_n$ in Levi's theorem. A similar result holds for decreasing sequences. Then as usual one derives Fatou's lemma and finally, Lebesgue's dominated convergence theorem:

(10) if $\varphi_n \in L$ and $\lim_n \varphi_n = \varphi$ (a.e.) and $|\varphi_n| \leq \psi$ where $\psi \in L$ then $\varphi \in L$ and $\int \varphi = \lim_n \int \varphi_n$.

In particular, it is a corollary that

(11) every bounded measurable function is in L .

Another useful result is that

(12) if $\varphi \in L$ and $\varphi \geq 0$ and $\int \varphi = 0$ then $\varphi(x) = 0$ (a.e.).

The theory of Lebesgue measurability can be recaptured by defining:

(13) (i) A is measurable if its characteristic function χ_A is measurable, and

(ii) $\mu(A) = \int \chi_A$.

From (12) one gets that A is of measure 0 just in case $\int \chi_A = 0$. The complete additivity of μ is derivable from Lebesgue's theorem (10). Finally, integration can be restricted to any measurable $A \subseteq I$ by taking

(14) $$\int_A \varphi = \int (\varphi \cdot \chi_A).$$

This has the same general properties as integration over I.

To generalize the theory of integration to all of $\mathbb{R}$ (and thence, any measurable subset of $\mathbb{R}$) one modifies the definition of S^+ by taking the class S to consist of those step functions which vanish outside of some finite interval.

For a generalization to $\mathbb{R}^p$, one replaces intervals by boxes $J = I_1 \times \ldots \times I_p$. Then step functions are defined to be those which assume constant values on a finite number of disjoint boxes and are otherwise 0. The reduction of the resulting integral to iterated integration is accomplished by Fubini's theorem. All this is carried out by the same limited means as used above (in $\mathrm{VFT}{\upharpoonright} + (\mu)$).

4.4 Banach spaces and Hilbert spaces. We assume familiarity here with a certain amount of introductory functional analysis. The following takes Kreyszig 1978 as principal source. The basic notion is that of a normed vector space over a field K; throughout here K is $\mathbb{R}$ or $\mathbb{C}$. The norm is a map $x \mapsto \|x\|$ from A into $\mathbb{R}$ satisfying

(1) (i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|ax\| = |a| \cdot \|x\|$

(iii) $\|x+y\| \leq \|x\| + \|y\|$.

The equality $x = y$ between elements of X is here some equality relation $=_X$, not necessarily the identity relation. (The map is sometimes called a <u>semi-norm</u> in these conditions.) A metric is then determined on X by

(2) $$d(x,y) = \|x-y\| .$$

A <u>Banach space</u> is a normed vector space which is complete in its metric.

The following are examples:

(3) (i) $\mathbb{R}^n$ and $\mathbb{C}^n$ <u>are Banach spaces using</u> $\|x\| = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$ <u>for</u> $x = (x_1,\ldots,x_n)$.

(ii) <u>For</u> $p \geq 1$, <u>the set</u> ℓ^p <u>of sequences</u> $x = \langle x_n \rangle$ <u>such that</u> $\Sigma\, |x_n|^p$ <u>converges is a Banach space with the norm</u> $\|x\| = (\Sigma\, |x_n|^p)^{1/p}$.

(iii) $C(I, \mathbb{R})$, <u>for</u> I <u>a finite interval in</u> $\mathbb{R}$, <u>is a Banach space with norm</u> $\|f\| = \sup_{x \in I} |f(x)|$.

All of these are <u>separable</u> Banach spaces. An example of an inseparable space is ℓ^∞ , The set of bounded sequences with norm $\sup_n |x_n|$. A very important class of examples is provided by L^p for any $p \geq 1$. This is defined to consist of all φ such that $\int |\varphi|^p$ exists (as a finite number). The norm is here defined by

(4) $$\|\varphi\| = (\int |\varphi|^p)^{1/p} .$$

Equality on L^p is defined by $\varphi = \psi \Leftrightarrow \|\varphi - \psi\| = 0$. The Hölder and Minkowski inequalities show that L^p is a normed space over $\mathbb{R}$. The Riesz-Fischer theorem tells us that L^p is complete and hence that

(5) L^p <u>is a Banach space</u>.

The proof of completeness goes as follows. Suppose given a sequence $\langle\varphi_n\rangle$ which is Cauchy in the metric of L^p. First choose n_o large enough so that $\|\varphi_n - \varphi_m\| < 1/2$ for all $n, m \geq n_o$. Having defined $n_1, \ldots, n_k$, choose $n_{k+1} > n_k$ so large that $\|\varphi_n - \varphi_m\| < 1/2^{k+2}$ for all $n, m \geq n_{k+1}$. Let $\tilde{\varphi}_k = \varphi_{n_k}$. If it is shown that the subsequence $\langle\tilde{\varphi}_k\rangle$ converges to an element of L^p, then the same holds for the original sequence. Now let $\psi_k = |\tilde{\varphi}_{k+1} - \tilde{\varphi}_k|$. Then $\|\psi_k\| < 1/2^{k+1}$ and so

$$\|\psi_o + \ldots + \psi_k\| < 1/2 + \ldots + 1/2^{k+1} < 1 .$$

Let $\theta_n = \Sigma_{k=0}^n \psi_k$. Then $\theta_n \nearrow$ and $\int \theta_n \leq (\int \theta_n^p)^{1/p} = \|\theta_n\|$ by Hölder's inequality and so $\int \theta_n$ is uniformly bounded by 1. By the monotone convergence theorem, $\Sigma_{k=1}^\infty \psi_k$ exists a.e. and is integrable. Thus for almost all x, $\Sigma_{k=1}^\infty |\tilde{\varphi}_{k+1}(x) - \tilde{\varphi}_k(x)|$ exists, so the same holds for

$\Sigma_{k=1}^{\infty}(\tilde{\varphi}_{k+1}(x) - \tilde{\varphi}_k(x))$. But $\Sigma_{k=1}^{p}(\tilde{\varphi}_{k+1}(x) - \tilde{\varphi}_k(x)) = \tilde{\varphi}_{p+1}(x) - \tilde{\varphi}_1(x)$, which shows that $\lim_{p \to \infty} \tilde{\varphi}_p(x) = \varphi(x)$ exists a.e. From this one may proceed to show that $\varphi(x)$ is the limit of the sequence $\tilde{\varphi}_p$ in the metric of L^p.

It is to be noted that L^p is also separable, since each of its members is a limit of rational-valued step functions which jump at rational points.

An <u>inner product space</u> is a vector space with a binary operation $x,y \mapsto \langle x,y \rangle$ satisfying

(6) (i) $\langle x,y \rangle$ is linear in x

(ii) $\langle x,x \rangle \geq 0$ <u>and</u> $\langle x,x \rangle = 0 \Leftrightarrow x = 0$

(iii) $\langle x,y \rangle = \overline{\langle y,x \rangle}$.

In (iii) $\bar{z}$ is the complex conjugate of z. If we are dealing with spaces over $\mathbb{R}$ we can simply take $\langle x,y \rangle = \langle y,x \rangle$. Every inner product space becomes a normed space using

(7) $$\|x\| = \sqrt{\langle x,x \rangle} \quad .$$

A <u>Hilbert space</u> is a complete inner product space.

The following are examples:

(8) (i) $\mathbb{R}^n$ <u>and</u> $\mathbb{C}^n$ <u>are Hilbert spaces with</u> $\langle x,y \rangle = \Sigma_{i=1}^{n} x_i \bar{y}_i$

(ii) ℓ^2 <u>is a Hilbert space with</u> $\langle x,y \rangle = \Sigma_i x_i \bar{y}_i$

(iii) L^2 <u>is a Hilbert space with</u> $\langle \varphi, \psi \rangle = \int \varphi \bar{\psi}$.

In this last we can take L^2 to consist of the complex valued functions such that $\int|\varphi|^2$ exists. All of these spaces are separable.

<u>Throughout the following, all Banach spaces and Hilbert spaces are assumed to be separable</u>. This hypothesis is not explicitly mentioned.

In a Hilbert space H we say x, y are orthogonal and write $x \perp y$ if $\langle x,y\rangle = 0$. We write $x \perp Y$ if $x \perp y$ for each $y \in Y$. Then the orthogonal complement of Y is defined by

(9) $$Y^{\perp} = \{x \mid x \perp Y\} .$$

A subset Y of H is said to be <u>convex</u> if whenever $0 \leq a \leq 1$ and $y_1, y_2 \in Y$ then $ay_1 + (1-a)y_2 \in Y$. Obviously every subspace of H is convex. Y is complete if it is closed under limits of Cauchy sequences from Y, i.e. if $\bar{Y}^{(s)} \subseteq Y$. It is a standard theorem that for each complete convex subset Y of H and each $x \in H$ there is a unique point y of Y which is closest to x. The proof starts by taking $d = \inf \|x-y\|$. We can only prove here that d exists under the additional hypothesis of separability:

(10) <u>if</u> Y <u>is a complete separable convex subset of</u> H <u>then for each</u> $x \in H$ <u>there is a unique element</u> y <u>of</u> Y <u>with</u> $\|x-y\|$ <u>a minimum</u>.

For the proof, let $\{u_n\}$ be a dense denumerable subset of Y and let $d_n = \|x-u_n\|$. Choose a subsequence $\langle n_k\rangle$ such that $\lim_k d_{n_k} = d = \inf_n d_n$.

It may be shown that $\langle u_{n_k} \rangle$ is Cauchy so its limit $y \in Y$, and $\|x-y\| = d = \inf_{u \in Y} \|x-u\|$. Convexity is used to prove that y is unique.

Y is called <u>located</u> if it satisfies the conclusion of (11), i.e. for any $x \in H$ we can find $y \in Y$ with $\|x-y\| = \inf_{u \in Y} \|x-u\|$. If Y is convex, this y is unique.

(11) <u>if</u> Y <u>is a convex located subset of</u> H <u>then for each</u> $x \in H$, <u>the</u> y <u>in</u> Y <u>which is closest to</u> x <u>satisfies</u> $(x-y) \perp Y$.

For otherwise we can find $y_1 \in Y$ with $\langle z, y_1 \rangle = b \neq 0$. Given any a we have

$$\|z-a y_1\|^2 = \langle z-a y_1, z-ay_1 \rangle = \langle z,z \rangle - \bar{a}\, b - a(\bar{b} - \bar{a}\langle y_1, y_1 \rangle).$$

Taking $a = \dfrac{b}{\|y_1\|^2} = \dfrac{b}{\langle y_1, y_1 \rangle}$ we get $\|z-a y_1\|^2 = \|z\|^2 - \dfrac{|b|^2}{\|y_1\|^2} < \|z\|^2$,

which would give $z-ay_1 = x-y-ay_1 = x-y_2$ with $y_2 \in Y$ closer than y. Putting (10), (11) together we obtain:

(12) <u>if</u> Y <u>is a complete located subspace of</u> H <u>then</u> $H = Y \oplus Y^{\perp}$.

Now for any countably generated subspace Z of H, the strong closure $\bar{Z}^{(s)}$ is a complete subspace of H so we can apply (12).

(13) <u>If</u> Z <u>is a countably generated subspace of</u> H <u>then</u> Z <u>is dense in</u> H <u>if and only if</u> $Z^{\perp}$ <u>consists just of</u> 0.

For suppose Z is dense in H. Given $x \in Z^{\perp}$ we can find a sequence $x_n \in Z$ which approaches x. Each $x_n \perp x$ so $\langle x_n, x\rangle \to \langle x,x\rangle$ and thus $\|x\|^2 = 0$ and $x = 0$. Conversely suppose $Z^{\perp} = \{0\}$. Let $Y = \bar{Z}^{(s)}$. Then $H = Y \oplus Y^{\perp}$; but $Y^{\perp} \subseteq Z^{\perp}$ so $H = Y$, and Z is dense in H.

(14) H <u>has an orthonormal basis</u>.

That is, we can find a sequence $\langle e_n \rangle$ such that $\|e_n\| = 1$ and $e_n \perp e_m$ for $n \neq m$ and such that the subspace generated by the e_n's is dense in H. This is found by the usual Gram-Schmidt process of orthonormalization applied to any dense denumerable subset of H.

4.5 <u>Linear operators and functionals</u>. If A_1, A_2 are two Banach spaces we may consider the totality of <u>linear operators</u> $T : A_1 \to A_2$. Of special interest are those T which are <u>continuous</u>. As usual, this is shown equivalent to the existence of a constant M such that

$$\|Tx\| \leq M\|x\| \tag{1}$$

for all $x \in A_1$. Thus T is called <u>bounded</u> in this case, and the collection of all bounded operators from A_1 to A_2 is denoted by $\mathcal{B}(A_1, A_2)$. Now $\mathcal{B}(A_1, A_2)$ is itself a vector space. To make it into a normed vector space we take

$$\|T\| = \inf_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \inf_{\|x\| = 1} \|Tx\| . \tag{2}$$

This is the same as $\inf_n \frac{\|Tu_n\|}{\|u_n\|}$ where $\{u_n\}$ is a dense denumerable subset of A_1, and so $\|T\|$ is well defined.

(3) $\mathcal{B}(A_1, A_2)$ is a Banach space.

To prove completeness, given a Cauchy sequence $\langle T_n \rangle$ one defines $Tx = \lim_n T_n x$ and shows $T \in \mathcal{B}(A_1, A_2)$ and $\lim_n \|T_n - T\| = 0$. (Actually, only A_2 need be assumed complete for this.) When $A_1 = A_2 = A$ we write $\mathcal{B}(A)$ for $\mathcal{B}(A,A)$; its elements are called the bounded operators on A.

If A is a Banach space, then the linear functionals on A are the linear operators $T : A \to K$ where K is the field of scalars of A, which is either $\mathbb{R}$ or $\mathbb{C}$. Then the bounded linear functionals are just the members of $\mathcal{B}(A,K)$, which is called the dual space of A and is denoted A^*. The letters $f, g, \ldots,$ are used to range over A^*.

If H is a Hilbert space then for each $z \in H$ we obtain a linear functional f by

(4) $$f(x) = \langle x, z \rangle .$$

The Riesz representation theorem tells us conversely that every element of H^* can be represented in this form. If $f = 0$ this is trivial. We may assume $f \neq 0$. Let $N(f) = \{x | f(x) = 0\}$ be the null space of f. z will be chosen perpendicular to $N(f)$ with $z \neq 0$. To do this we wish to get a decomposition $H = N(f) \oplus N(f)^{\perp}$; then $N(f)^{\perp} \neq \{0\}$, for otherwise

$H = N(f)$ and $f = 0$. To apply the preceding section, it is evident that $N(f)$ is complete; we have to show that it is located. Now $N(f)$ is not obviously separable, but for any $m > 0$ we can form $Y_m = \{x | \|f(x)\| < 1/m\} \supseteq N(f)$ which is separable, since for each x with $f(x) = 0$ we can find u_n so that $\|f(u_n)\| < 1/m$. Further Y_m is easily seen to be convex. Hence by 4.4(10) we can find a unique $y_m \in Y_m$ which is closest to x. Now let $d_m = \|x-y_m\|$. For each $z \in N(f)$ we have $d_m \leq \|x-z\|$ since $N(f) \subseteq Y_m$. Thus the d_m's are bounded above and $d = \sup_m d_m = \lim_m d_m$ exists (d_m is increasing). Now for any $z \in N(f)$ we have $d \leq \|x-z\|$. It cannot be that $\|x-z\| < d$ for some $z \in N(f)$, for otherwise $\|x-z\| < d_m$ for sufficiently large m. Take a neighborhood S of z such that $S \subseteq Y_m$. Then we can choose u_n in S so close to z that $\|x-u_n\| < d_m$ contradicting the definition of d_m. Returning to y_m, the closest element of Y_m to x, we now show the sequence $\langle y_m \rangle$ is Cauchy. For let $z_m = x-y_m$. Then $\|z_m\| = d_m$ and $\|z_m + z_p\| = \|2x - (y_m + y_p)\| = 2\|x - \frac{(y_m + y_p)}{2}\|$. If $m \geq p$ then $\frac{y_m + y_p}{2} \in Y_m$ because it is convex, so $\|z_m + z_p\| \geq 2\, d_m$ and $\|y_m - y_p\|^2 = \|z_m - z_p\|^2 = -(\|z_m + z_p\|^2) + 2(\|z_m\|^2 + \|z_p\|^2)$ by the parallelogram law so $\|y_m - y_p\|^2 \leq -(2\, d_m)^2 + 2(d_m^2 + d_p^2)$. But as $m, p \to \infty$ we have $d_m \to d$, $d_p \to d$, so the r.h.s. $\to 0$ and $\|y_m - y_p\| \to 0$. (The parallelogram law says that $\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2)$.) Let $y = \lim_m y_m$. Then $f(y) = \lim_m f(y_m)$ so $\|f(y)\| = \lim_m \|f(y_m)\| = 0$. Hence $f(y) = 0$ and $y \in N(f)$. Finally $\|x-y\| = \lim_m \|x-y_m\| = \lim_m d_m = d$, so y is the closest element of $N(f)$ to x. We have thus proved that $N(f)$ is located. Returning to our argument above it follows that $H = N(f) \oplus N(f)^{\perp}$ and that $N(f)^{\perp} \neq \{0\}$. The proof of Riesz's theorem is now carried out in a standard way. Pick $z_o \neq 0$ with $z_o \perp N(f)$, and let $v = f(x)z_o - f(z_o)x$ for any $x \in H$. Then $f(v) = f(x)f(z_o) - f(z_o)f(x) = 0$ so $v \in N(f)$. Hence $\langle v, z_o \rangle = 0$, which shows $f(x)\langle z_o, z_o\rangle - f(z_o)\langle x, z_o\rangle = 0$. Then

$$f(x) = \frac{f(z_o)}{\|z_o\|^2} \langle x, z_o \rangle .$$

To satisfy (4) it is thus sufficient to take $z = \frac{\overline{f(z_o)}}{\|z_o\|^2} z_o$.

(5) <u>For each linear functional</u> f <u>on</u> H <u>there is a unique</u> z <u>with</u> $f(x) = \langle x, z \rangle$ <u>for all</u> x.

Unicity is direct: if $f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle$ for all x then $\langle x, z_1 - z_2 \rangle = 0$ and so for $x = (z_1 - z_2)$ we have $\|z_1 - z_2\|^2 = 0$ and hence $z_1 - z_2 = 0$. The following is also easy.

(6) <u>If</u> f <u>is bounded then for the unique</u> z <u>in</u> (5), $\|f\| = \|z\|$.

Given any operator $T : H \to H$ consider for each $y \in H$ the function

(7) $$f(x) = \langle Tx, y \rangle .$$

This is seen to be linear and bounded. By the Riesz representation theorem, there is a unique z, denoted T^*y, such that $\langle Tx,y\rangle = \langle x,z\rangle$; i.e.

$$(8) \qquad \langle Tx,y\rangle = \langle x, T^*y\rangle$$

for all x. It may be seen directly that

$$(9) \qquad \underline{\text{if}}\ \ T \in \mathcal{B}(H)\ \ \underline{\text{then}}\ \ T^* \in \mathcal{B}(H)\ \ \underline{\text{and}}\ \ \|T^*\| = \|T\| .$$

T^* is called the operator which is <u>adjoint to</u> T. If $T = T^*$ then T is said to be <u>self-adjoint</u> (<u>or Hermitian</u>). A more general class of operators whose theory is well developed are the <u>normal operators</u>, defined by the condition $TT^* = T^*T$. An operator is said to be <u>unitary</u> if it is normal and $TT^* = I$, i.e. $T^* = T^{-1}$; these are the same as the <u>isometric</u> operators, i.e. for which $\langle Tx, Ty\rangle = \langle x,y\rangle$.

Adjoint operators cannot be defined by (8) in Banach spaces A which are not Hilbert spaces. Instead one associates in a natural way the operator $T^* : A^* \to A^*$ on the space dual to A given by

$$(10) \qquad (T^*(f))(x) = f(Tx).$$

This is just another form of (8) when A is a Hilbert space, using the Riesz representation theorem for any f in A^*. Adjoints can be defined more generally for any $T \in \mathcal{B}(A_1, A_2)$.

To return to linear functionals on any normed space, we now obtain a form of the Hahn-Banach theorem for these. The usual result requires the axiom of choice (in the form of Zorn's lemma) for its proof. However this is not necessary when dealing with separable spaces.

Throughout the following, A is assumed to be a separable normed space and p is assumed to be a continuous seminorm, i.e.

(11) (i) $p : A \to \mathbb{R}$, $p(x) \geq 0$

(ii) $p(x+y) \leq p(x) + p(y)$,

(iii) $p(ax) = |a|\, p(x)$ for any scalar a, and

(iv) p is continuous.

One first establishes the following lemma for real spaces.

(12) If X is a separable subspace of A and f is a linear functional on X satisfying $|f(x)| \leq p(x)$ for all $x \in X$, then for each $z \in A - X$ we can find an extension g of f to the span $X + (z)$ of $X \cup \{z\}$ with $|g(y)| \leq p(y)$ for all $y \in X + (z)$.

For the proof, one first shows that each element of $X + (z)$ has a unique representation in the form $x + az$ where $x \in X$. g is defined by

(13) $$g(x + az) = f(x) + a\, c$$

for suitable choice of c. Now for any $x, y \in X$ we have

$$f(x-y) = f(x)-f(y) \leq p(x-y) = p(x+z-y-z) \leq p(x+z)+p(-y-z), \quad \text{so}$$

$$p(-y-z) - f(y) \leq p(x+z) - f(x).$$

It is sufficient to show $\sup_{y \in X} [p(-y-z)-f(y)]$ and $\inf_{x \in X} [p(x+z)-f(x)]$ exist, for then we can choose c inbetween; it is shown as usual that any such choice of c in (13) gives the desired conclusion in (12). Now since p is continuous and $p(0) = 0 \cdot p(0) = 0$, then f is continuous on X ($|f(x)| \leq p(x)$ implies $f(x)$ approaches 0 as x approaches 0). Thus $p(x+z) - f(x)$ is continuous as a function of $x \in X$ and $p(-y-z)-f(y)$ is continuous as a function of $y \in X$. Since X is separable the required sup's and inf's exist.

The Hahn-Banach theorem here takes the following form:

(14) Under the same hypotheses as in (11) and (12) we can find an extension $\tilde{f}$ of f to A satisfying $|\tilde{f}(x)| \leq p(x)$ for all $x \in A$.

One does the real case first. Fix $\{u_n\}$ as a dense subset of X and $\{w_n\}$ as a dense subset of A. Define a sequence of linear functionals g_k on subspaces X_k with $|g_k(x)| \leq p(x)$ as follows: $g_o = f_o$ and $X_o = X$. Given g_k, X_k and a dense basis for X_k, consider the least n, if any, with $w_n \notin X$. (If there is none we take $X_{k+1} = X_k$, $g_{k+1} = g_k$). Then $X_{k+1} = X_k + (w_n)$ and the basis of X_k is expanded by w_n. Now g_{k+1} is defined as an extension of g_k to X_{k+1} satisfying $|g_{k+1}(x)| \leq p(x)$ all

$x \in X_{k+1}$. Let $g = \bigcup_k g_k$ which is defined on $X = \bigcup_k X_k$. Since each w_n belongs to X, X is dense in A. Hence g may be extended to $\tilde{f}$ on A by continuity, and $|\tilde{f}(x)| \leq p(x)$ also holds on A by continuity of p. The complex case of the theorem is obtained from the real case as usual.

4.6 <u>The uniform boundedness and open mapping theorems</u>. These are fundamental results for Banach spaces and are both proved using the Baire category theorem. Throughout this section A is any separable Banach space and $\{u_n\}$ is a dense denumerable subset. The usual formulation of the Baire category theorem (for complete metric A) is that A <u>is not a countable union of nowhere dense sets.</u> X is called nowhere dense if its closure $\bar{X}$ contains no basic open neighborhood, i.e. if for each $x \in \bar{X}$ and $r > 0$ there exists $y \in A - \bar{X}$ with $y \in S(x; r)$. Classically, the complement of $\bar{X}$ is an open set G and this condition is the same as saying that G is dense. The form of Baire category theorem which was proved in 2.13 above is that

(1) <u>if $\langle G_n \rangle$ is a sequence of dense open sets then $\bigcap_n G_n$ is also dense</u>.

Hence in this case $\bigcup_n F_n \neq A$. (1) implies the classical version assuming that each closure $\bar{X}$ is the complement of an open set, but not in our restricted theory. We thus have to examine the arguments more carefully to see that (1) itself suffices. This is very easy for the first of the theorems considered, but takes more work for the second.

The uniform boundedness theorem is as follows.

(2) <u>If</u> $\langle T_n \rangle$ <u>is a sequence of bounded operators on</u> A <u>(to</u> A) <u>and if for each</u> $x \in A$ <u>there exists</u> M <u>such that</u> $\|T_n x\| \leq M$ <u>for all</u> n <u>then there exists</u> M <u>such that for all</u> n, $\|T_n\| \leq M$.

By hypothesis, $A = \bigcup_k X_k$ where $X_k = \{x \mid \forall n \, \| T_n x\| \leq k\}$, so

(3) $$\bigcap_k G_k = \Lambda \text{ where } G_k = \{x \mid \exists n (\|T_n x\| > k)\}.$$

We show that each G_k is open, i.e. is a countable union of open balls. Indeed, since each T_n is continuous, if $\|T_n x\| > k$ then for some $\delta > 0$ we have $\|T_n x\| \geq k + \delta$ and so for some $\epsilon > 0$ and all y in $\delta(x;\epsilon)$ we have $\|T_n y\| \geq k + \delta$; but this is equivalent to $\|T_n u_m\| \geq k + \delta$ for all basis elements $u_m \in S(x;\epsilon)$. From this we see that

(4) $$x \in G_k \Leftrightarrow \exists n \, \exists \delta > 0 \, \exists \rho > 0 \, \exists p [x \in S(u_p;\rho) \wedge \forall u_m \in S(u_p;\rho)(\| T_n u_m\| \geq k + \delta)].$$

The r.h.s. gives the desired representation of G_k as an open set. Hence

(5) <u>some</u> G_k <u>is not dense in</u> A ,

by our form of the Baire category theorem. The rest of the proof goes as usual: some X_k contains an open ball $S(x_o;r)$ with $r > 0$ so that

(6) $$y \in \delta(x_o;r) \Rightarrow \forall n(\|T_n\| \leq k).$$

For any $x \in X$ with $x \neq 0$, let $y = x_o + sx$ where $s = \frac{r}{2\|x\|}$. Then $\|y - x_o\| < r$ so $\|T_n y\| \leq k$ for all n. Also $\|T_n x_o\| \leq k$ for all k.

Now $x = \frac{1}{s}(y-x_o)$ and $\|T_n x\| = \frac{1}{s}\| T_n(y - x_o)\| \leq \frac{1}{s}(\|T_n y\| + \|T_n x_o\|) \leq \frac{2k}{s} = \frac{4k\|x\|}{r}$.

Hence $\|T_n\| \leq \frac{4k}{r}$, which is the desired uniform bound.

Next, in the open mapping theorem it is shown that if $T \in \mathcal{B}(A)$ is one-one and onto then T^{-1} is in $\mathcal{B}(A)$, too. Actually, we must also show here that T^{-1} exists as an operation. First a lemma is proved about arbitrary $T \in \mathcal{B}(A)$. Given any X we write $T(X)$ for $\{Tx | x \in X\}$, $X + w$ for $\{x + w | x \in X\}$ and aX for $\{ax | x \in X\}$. Then by linearity $T(X + w) = T(X)+Tw$ and $T(aX) = aT(X)$. Let $\{B_k\}$ be an enumeration of all open balls $S(u_{k_o}; r_{k_1})$ with center in the dense basis and rational radius, and let $S_n = S(0; 1/2^n)$. We first aim to show that

(7) <u>for each</u> m, $\overline{A - T(mS_1)}$ <u>is open</u>.

For this, we claim that

(8) <u>the following are equivalent</u>:

(i) $B_k \cap \overline{T(mS_1)} = 0$;

(ii) $B_k \cap T(mS_1) = 0$; <u>and</u>

(iii) <u>for each</u> $u_p \in mS_1$ <u>we have</u> $T(u_p) \notin B_k$.

For obviously (i) ⇒ (ii) ⇒ (iii). Also (ii) ⇒ (i) because if $y \in B_k \cap \overline{T(mS_1)}$ then there exists $z \in B_k \cap T(mS_1)$. Thus suppose (iii); we show (ii). Suppose contrary to (ii) that $y \in B_k \cap T(mS_1)$. Then $y = Tx$ where $x \in mS_1$. Since $y \in B_k$ we can find $\epsilon > 0$ such that $S(y; \epsilon) \subseteq B_k$. Then by continuity of T we can find $\delta > 0$ such that $T : S(x;\delta) \to S(y;\varepsilon)$. Also δ can be chosen so small that $S(x;\delta) \subseteq m S_1$. But there exists $u_p \in S(x;\delta)$ so we have found a $u_p \in m S_1$ with $T(u_p) \in S(y;\varepsilon) \subseteq B_k$, contrary to (iii).

Now by (8)(iii) we can decide whether $B_k \cap \overline{T(m S_1)} = 0$ and so

(9) $$G_m = \bigcup B_k[B_k \cap \overline{T(m S_1)} = 0] \text{ \underline{is} \underline{open}} .$$

To complete the proof of (7) we show that

(10) $$A - \overline{T(m S_1)} = G_m .$$

For obviously we have $\supseteq$. Conversely, suppose $y \notin \overline{T(mS_1)}$. Then for some B_k, $y \in B_k$ and $B_k \cap T(m S_1) = 0$. But then by (8) $B_k \cap \overline{T(m S_1)} = 0$, so $y \in G_m$.

We can now prove by the standard argument that

(11) <u>if</u> T <u>is</u> <u>surjective</u> <u>then</u> $T(S_o)$ <u>contains</u> <u>some</u> <u>open</u> <u>ball</u> <u>about</u> 0.

For, first, $A = \bigcup_{m=1}^{\infty} m S_1$ so $A = T(A) = \bigcup_{m=1}^{\infty} T(m S_1)$. Hence

(12) $$A = \bigcup_{m=1}^{\infty} \overline{T(m S_1)} \quad \text{and} \quad \bigcap_{m=1}^{\infty} G_m = \Lambda .$$

By our form of the Baire category theorem, there exists an m such that G_m is not dense and so there is an open ball $B_p = S(y_o ; r_o) \subseteq \overline{T(m S_1)}$. Now

$\overline{T(mS_1)} = \overline{mT(S_1)} = \overline{mT(S_1)}$, so $S(y_o; \frac{1}{m} r_o) \subseteq \overline{T(S_1)}$. Shifting back to 0 we get an $\epsilon > 0$ such that

(13) $$S(0;\epsilon) \subseteq \overline{T(S_1)} - y_o .$$

Next it is shown that

(14) $$\overline{T(S_1)} - y_o \subseteq \overline{T(S_o)} .$$

For this, consider any $y \in \overline{T(S_1)} - y_o$, i.e. $y + y_o \in \overline{T(S_1)}$; also $y_o \in \overline{T(S_1)}$. Given $r > 0$, choose $w, z \in S_1$ with $\|Tw - (y+y_o)\| < \frac{r}{2}$ and $\|Tz - y_o\| < \frac{r}{2}$. Then $\|Tw-Tz - y\| < r$, so $\|T(w-z)-y\| < r$. Also $\|w-z\| < 1$ so $(w-z) \in S_o$. We have thus found $u \in S(y;r)$ with $Tu \in T(S_1)$. Since this holds for each $r > 0$ it follows that $y \in \overline{T(S_o)}$. Now as a corollary of (13) and (14) we have

(15) $$S(0; \frac{\epsilon}{2^n}) \subseteq \overline{T(S(0; \frac{1}{2^n}))} = \overline{T(S_n)} \quad \text{for } n = 1,2,\ldots .$$

It is shown finally that

(16) $$S(0; \epsilon/2) \subseteq T(S_o).$$

Given $y \in S(0; \epsilon/2)$ we pick $u_{n_1} \in S_1$ with $\|y-Tu_{n_1}\| < \epsilon/4$ by (15). Having chosen $n_1,\ldots,n_k$ with $\|y-\Sigma_{i=1}^k Tu_{n_i}\| < \epsilon/2^{k+1}$ we pick $u_{n_{k+1}} \in S_{k+1}$ by (15) so that $\|y - \Sigma_{i=1}^k Tu_{n_i} - Tu_{n_{k+1}}\| < \epsilon/2^{k+1}$. If we put $z_k = \Sigma_{i=1}^k u_{n_i}$ we have $\|u_{n_i}\| < 1/2^i$ so for $k \leq \ell$, $\|z_k-z_\ell\| < \Sigma_{i=k+1}^\infty \frac{1}{2^i} = \frac{1}{2^k}$.

It follows that $\langle z_k \rangle$ is Cauchy and converges to some $x = \Sigma_{i=1}^{\infty} u_{n_i}$. Again $\|x\| \leq \Sigma_{i=1}^{\infty} \frac{1}{2^i} = 1$ so $x \in S_o$. Since $Tz_k \to y$, we have $Tx = y$ by continuity, completing the proof.

This proof also establishes the following.

(17) *If* T *is surjective then for each* $y \in T(S_o)$ *we can determine an element* $x \in S_o$ *such that* $Tx = y$.

Suppose now that T is one-one and onto; then x is uniquely determined by y and is denoted $T^{-1}y$. The open mapping theorem says that

(18) *if* T *is one-one and onto then* T^{-1} *is continuous* ,

or what comes to the same thing, that if G is open then $T(G)$ is open. Given $x \in G$, pick $r > 0$ with $S(x;r) \subseteq G$, so $S(0;r) \subseteq G-x$. Let $k = \frac{1}{r}$. Then $k(G-x)$ contains $S(0;1) = S_o$. By the lemma (11), $T(k(G-x)) = k(TG-Tx)$ contains an open ball $S(0;\epsilon)$ about 0; going back, $T(G) - Tx$ also contains the open ball $S(0;r\epsilon)$ so $T(G)$ contains the open ball $S(Tx;r\epsilon)$. To do this explicitly we need only have a method to choose ϵ satisfying (13), i.e. for which $S(y_o;\epsilon) \subseteq \overline{T(S_1)}$; but this is equivalent to $\forall u_p \in S(y_o;\epsilon)[u_p \in \overline{T(S_1)}]$. Thus we can find a representation for $T(G)$ as an open set in terms of one for G as an open set, and the theorem (18) is proved.

4.7 Spectral theory of compact self-adjoint operators. An operator T on a Banach space is said to be compact (or completely continuous) if we can associate with each bounded sequence $\langle x_n \rangle$ a convergent subsequence of $\langle Tx_n \rangle$. Every compact operator is bounded. The spectral theory of compact operators goes quite far without additional assumptions. For simplicity we restrict attention to such operators on a Hilbert space over $\mathbb{C}$. The classic examples are from the theory of integral equations. Given a bounded real interval $I = [a,b]$ one takes H to be $C(I,\mathbb{C})$ or to be the complex-valued functions in $L^2(I)$ and takes

$$(1) \qquad (Tf)(x) = \int_a^b K(x,y)f(y)dy \, .$$

This is a compact operator if K is in $C(I \times I, \mathbb{C})$ or $L^2(I \times I)$, resp. If $K(x,y) = \overline{K(y,x)}$ then T is self-adjoint.

Assume T is any compact operator on H. An eigenvalue λ of T is a complex number such that for some $x \neq 0$, $(T-\lambda I)x = 0$, i.e. $Tx = \lambda x$; any such x is called an eigenvector for the eigenvalue λ. The space spanned by these is just $N(T-\lambda I)$. By the same argument as for the Riesz representation theorem in 4.5 we get

$$(2) \qquad H = N(T-\lambda I) \oplus N(T-\lambda I)^{\perp} \, .$$

Now any vector $u \in H$ has a unique decomposition $x = v + w$ into these two subspaces. If we let u run through a dense denumerable set $\{u_n\}$, we obtain a dense denumerable subset $\{v_n\}$ of $N(T-\lambda I)$. Then by the Gram-Schmidt process we obtain an orthonormal basis for $N(T-\lambda I)$. This basis may be infinite. However, we have:

(3) *if* T *is a compact operator then* $N(T-\lambda I)$ *has a finite basis.*

For suppose the orthonormal basis $x_0, \ldots, x_n, \ldots$ is infinite. Then for $n \neq m$ we have $\|Tx_n - Tx_m\|^2 = \|\lambda x_n - \lambda x_m\|^2 = |\lambda|^2 \|x_n - x_m\|^2 = |\lambda|^2 \langle x_n - x_m, x_n - x_m \rangle = 2|\lambda|^2$. Then no subsequence of $\langle Tx_n \rangle$ can be Cauchy, contrary to compactness.

Next we consider self-adjoint operators.

(4) *If* T *is a self-adjoint operator on* H *then:*

(i) *each eigenvalue of* T *is real;*

(ii) *eigenvectors corresponding to different eigenvalues are orthogonal,*

(iii) $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$; *and*

(iv) *each eigenvalue* λ *of* T *has* $|\lambda| \leq \|T\|$.

For (i), if X is an eigenvector for the value λ, then

$$\lambda\langle x,x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \bar{\lambda}\langle x,x\rangle ,$$

so $\lambda = \bar{\lambda}$ since $\langle x,x\rangle = \|x\|^2 \neq 0$. In (ii), if $Tx = \lambda x$, $Ty = \mu y$ with $x \neq 0$, $y \neq 0$ then λ, μ are real and

$$\lambda \langle x,y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \mu \langle x,y\rangle .$$

Thus if $\lambda \neq \mu$ then $\langle x,y\rangle = 0$. Now to prove (iii), first given any $\|x\|$ we have $|\langle Tx, x\rangle| \leq \|Tx\| \cdot \|x\|$ by Schwarz's inequality, so if $\|x\| = 1$,

$|\langle Tx,x\rangle| \leq \|T\|$. Let $M = \sup_{\|x\|=1} |\langle Tx,x\rangle|$. Then for any x, $|\langle Tx,x\rangle| \leq M\|x\|^2$ as seen by dividing x by $\|x\|$ when $x \neq 0$. Now for the reverse inequality consider any positive real k and any x with $\|x\| = 1$. To show $\|Tx\| \leq M$ we may assume $\|Tx\| \neq 0$. Let $x_1 = (kx + k^{-1}Tx)$, and $x_2 = (kx - k^{-1}Tx)$. Then

$$|\langle Tx_1,x_1\rangle| \leq M\|x_1\|^2 \quad \text{and} \quad |\langle Tx_2,x_2\rangle| \leq M\|x_2\|^2$$

so

$$|\langle Tx_1,x_1\rangle - \langle Tx_2,x_2\rangle| \leq M(\|x_1\|^2 + \|x_2\|^2).$$

The left hand side reduces by calculation to $4\|Tx\|^2$ using self-adjointness of T. The right hand side reduces to $2M(k^2\|x\|^2 + k^{-2}\|Tx\|^2)$ by the parallelogram law. Thus with $\|x\| = 1$ we have

$$4\|Tx\|^2 \leq 2M(k^2 + k^{-2}\|Tx\|^2).$$

Now the r.h.s. of the last inequality is minimized by taking $k = \|Tx\|^{1/2}$. Thus $4\|Tx\|^2 \leq 4M\|Tx\|$ and $\|Tx\| \leq M$, as was to be proved. Finally, to prove (iv) suppose $Tx = \lambda x$ with $x \neq 0$. Then $(T-\lambda I)x = 0$ so $(T-\lambda I)^{-1}$ cannot exist. On the other hand if $\lambda > \|T\|$ then the inverse of $(T-\lambda I)$ can be obtained as

$$(T-\lambda I)^{-1} = -\frac{1}{\lambda}(I-S)^{-1} = -\frac{1}{\lambda}[\, I + S + S^2 + \dots \,]$$

where $S = \frac{1}{\lambda}T$. We have $\|S\| = \frac{1}{\lambda}\|T\| < 1$ and $\|I+S+\dots+S^n\| \leq 1+\|S\|+\dots+\|S^n\|$, which shows that $I+S+S^2+\dots$ converges to an operator $\tilde{S}$; by check $(I-S)\tilde{S} = \tilde{S}(I-S) = I$.

(5) *If T is a compact self-adjoint operator then we can find an eigenvalue λ of T with $|\lambda| = \|T\|$.*

For, $\|T\| = \sup_{\|x\| \leq 1} |\langle Tx,x\rangle|$ so using a countable dense subset $\{w_n\}$ of $\{x \mid \|x\| \leq 1\}$, we can choose a sequence $\langle x_n\rangle$ with $\|x_n\| \leq 1$ and $\|x_n\| \to 1$ such that $|\langle Tx_n,x_n\rangle| \to \|T\|$. We can further take $\langle x_n\rangle$ so that $\langle Tx_n,x_n\rangle$ converges, so either $\langle Tx_n,x_n\rangle \to \|T\|$ or $\langle Tx_n,x_n\rangle \to -\|T\|$. Let $\lambda = \pm\|T\|$ according to which of these holds. Now

$$\|Tx_n - \lambda x_n\|^2 = \|Tx_n\|^2 - 2\lambda\langle Tx_n,x_n\rangle + \lambda^2\|x_n\|^2 \leq \|T\|^2 + \lambda^2 - 2\lambda\langle Tx_n,x_n\rangle$$

$$= 2\lambda^2 - 2\lambda\langle Tx_n,x_n\rangle \to 0.$$

Hence $(Tx_n - \lambda x_n) \to 0$. By compactness of T we can choose a subsequence $\langle x_{n_k}\rangle$ of the x_n's for which $\langle Tx_{n_k}\rangle$ converges. Then $\langle \lambda x_{n_k}\rangle$ also converges to λx for some x. (We may assume $\lambda \neq 0$, otherwise we have the trivial case of $T = 0$.) We conclude that $Tx = \lambda x$; further $\|x\| = 1$ since we chose $\|x_n\| \to 1$.

(6) *If T is a compact self-adjoint operator then we can find a sequence of eigenvalues $|\lambda_0| > |\lambda_1| > \ldots > |\lambda_n| > \ldots$ such that $\lim_n \lambda_n = 0$ and each eigenvalue λ has $|\lambda| = |\lambda_n|$ for some n.*

$|\lambda_0| = \|T\|$. Let $X_0 = N(T - \lambda_0 I)$ and $X_0^- = N(T + \lambda_0 I)$, both of which are finite-dimensional by (3) and orthogonal to each other by (4)(ii). We decompose H

as $X_o \oplus X_o^- \oplus H_1$ and consider the restriction T_1 of T to H_1. T_1 is again compact and self-adjoint and $\|T_1\| \leq \|T\|$. Then we have an eigenvalue λ_1 for T_1 with $|\lambda_1| = \|T_1\|$. Since $\lambda_1 \neq \pm \lambda_o$ we must have $|\lambda_1| < |\lambda_o|$. Proceeding in this way we obtain the sequence λ_n. Let $c = \lim_n |\lambda_n|$. We can find for each n an eigenvector x_n of $\pm \lambda_n$ with $\|x_n\| = 1$; for $|\lambda_n| \neq |\lambda_m|$ we have $\langle x_n, x_m \rangle = 0$. Then

$$\|Tx_n - Tx_m\|^2 = \|\pm \lambda_n x_n \mp \lambda_m x_m\|^2 = \lambda_n^2 + \lambda_m^2 \geq 2c^2 .$$

Hence if $c > 0$, the sequence $\langle Tx_n \rangle$ cannot converge, which is contrary to the compactness of T.

From here it is a direct step to the <u>spectral representation</u> of T. The Fredholm theory of integral equations is then derived as a special case.

5 <u>Uses of the axiom</u> Proj_1 <u>of type</u> 1 <u>projection.</u> This is a weak consequence of $(\exists^{\mathbb{N} \to \mathbb{N}})$ which was introduced in I. 4.8. We explain it again and then draw some consequences which cannot be obtained in $\mathrm{VFT} + (\mu)$, particularly the l.u.b. axiom for sets of reals.

First we consider statements $\mathrm{Proj}_{\mathbb{A}}^{\mathbb{B}}$ (for each specific pair of classes $\mathbb{A}$, $\mathbb{B}$) of which Proj_1 is a special case. These express that if $b \in \mathbb{S}(\mathbb{A} \times \mathbb{B})$ is any relation between elements of $\mathbb{A}$ and $\mathbb{B}$ then the projection of b along $\mathbb{B}$ exists as a subset $\underline{a}$ of $\mathbb{A}$:

$$(\mathrm{Proj}_{\mathbb{A}}^{\mathbb{B}}) \qquad \forall b \in \mathbb{S}(\mathbb{A} \times \mathbb{B}) \exists a \in \mathbb{S}(\mathbb{A}) \forall x^{\mathbb{A}} [x \in a \Leftrightarrow \exists y^{\mathbb{B}} (x,y) \in b].$$

Thus Proj_1 is the same as $\mathrm{Proj}_{\mathbb{N}}^{\mathbb{N} \to \mathbb{N}}$. In this case we can think of the given b as a sequence $\langle b_y \rangle_{y \in \mathbb{N} \to \mathbb{N}}$ of subsets of $\mathbb{N}$ and of $\underline{a}$ as $\bigcup b_y [y \in \mathbb{N} \to \mathbb{N}]$.

Recall from I.4.2 that $(\exists^{\mathbb{B}})$ is the axiom asserting the existence of a function $\exists^{\mathbb{B}} \in (\mathbb{S}(\mathbb{B}) \to \{0,1\})$ such that for any subset c of $\mathbb{B}$, $\exists^{\mathbb{B}}(c) = 0 \Leftrightarrow \exists y^{\mathbb{B}} (y \in c)$. It is immediate that $(\exists^{\mathbb{B}})$ implies $(\mathrm{Proj}_{\mathbb{A}}^{\mathbb{B}})$ for each $\mathbb{A}$; however, the converse is not true. We may express the difference by saying that in $(\mathrm{Proj}_{\mathbb{A}}^{\mathbb{B}})$, the set $\underline{a}$ is merely asserted to exist, given b, while from $\exists^{\mathbb{B}}$ we obtain $\underline{a}$ uniformly as a function of b.

To apply Proj_1 to get the l.u.b. axiom, it is more convenient to consider the representation of reals by the class $\mathbb{D}$ of Dedekind sections.

In the following 'r', 's' range over $\mathbb{Q}$ and $\langle r_n \rangle$ is a standard enumeration of $\mathbb{Q}$. $\mathbb{D}$ is defined as follows:

$$d \in \mathbb{D} \Leftrightarrow d \in \mathbb{S}(\mathbb{Q}) \wedge d \neq \Lambda \wedge \mathbb{Q} - d \neq \Lambda \wedge$$
$$\forall r,s[\, r < s \wedge s \in d \Rightarrow r \in d\,]$$
$$\wedge \neg \exists r[\, r \in d \wedge \forall s(s \in d \Rightarrow s \leq r)\,].$$

Each $x \in \mathbb{R}$ determines a Dedekind section $\overleftarrow{x} = (-\infty, x) \cap \mathbb{Q} = \{r : r < x\}$. Conversely, each Dedekind section d determines $x \in \mathbb{R}$ with $d = \overleftarrow{x}$ as follows. For each $n \in \mathbb{N}$ consider $\{m/2^n : m \in \mathbb{Z}\}$. Every rational lies between $m/2^n$ and $(m+1)/2^n$ for some n. Thus there are $m < m'$ with $m/2^n \in d$ and $m'/2^n \notin d$. If we take the least m' with $m'/2^n \notin d$ we have $m' = m+1$ where $m/2^n \in d$. (This uses the least element principle for upper intervals in $\mathbb{Z}$, which reduces to set-induction for $\mathbb{N}$.) m is uniquely determined and we put $r_n = m/2^n$. The sequence $x = \langle r_n \rangle$ is easily verified to be Cauchy and $d = \overleftarrow{x}$. The correspondence $x \mapsto \overleftarrow{x}$ thus makes $\mathbb{R} \cong \mathbb{D}$.

Next observe that $\mathrm{Proj}^{\mathbb{R}}_{\mathbb{N}}$ follows from Proj_1, i.e. $\mathrm{Proj}^{\mathbb{N} \xrightarrow{\cdot} \mathbb{N}}_{\mathbb{N}}$. For this we use a surjection v of $\mathbb{N} \xrightarrow{\cdot} \mathbb{N}$ on $\mathbb{R}$, e.g. $\langle n_i \rangle$ is sent onto $z_{n_0} + \sum_{i=1}^{\infty} sg(n_i)/2^i$, where $\langle z_k \rangle_{k \in \mathbb{N}}$ is an enumeration of $\mathbb{Z}$ and $sg(0) = 0$, $sg(n+1) = 1$. The real number assigned to each $g \in \mathbb{N} \xrightarrow{\cdot} \mathbb{N}$ in this way is denoted $v(g)$. Then if $b \subseteq \mathbb{N} \times \mathbb{R}$ we obtain $a \subseteq \mathbb{N}$ with

$$\forall n[\, n \in a \Leftrightarrow \exists x^{\mathbb{R}}(n,x) \in b\,]$$

from

$$\forall n[\, n \in a \Leftrightarrow \exists g^{\mathbb{N} \xrightarrow{\cdot} \mathbb{N}}(n, v(g)) \in b\,].$$

Furthermore $\text{Proj}^{\mathbb{R}}_{\mathbb{Q}}$ follows from $\text{Proj}^{\mathbb{R}}_{\mathbb{N}}$ by the enumeration of $\mathbb{Q}$.

If c is a non-empty set of reals which is bounded above, then the Dedekind section d of $\sup c$ would be defined by

$$r \in d \Leftrightarrow \exists x^{\mathbb{R}} (x \in c \wedge r < x).$$

From the preceding we may thus conclude in $\text{VFT}\restriction + (\mu) + (\text{Proj}_1)$ that every non-empty set of reals which is bounded above has a least upper bound. It follows directly that every non-empty set of reals which is bounded below has a g.l.b.

Another consequence of Proj_1 is that every set c of reals is open when each point of c is interior, i.e. when $\forall x^{\mathbb{R}} [x \in c \Rightarrow \exists n,m (x \in (r_n, r_m) \subseteq c)]$. For, given c the set $c^* = \{(n,m) | (r_n, r_m) \not\subseteq c\}$, or

$$(n,m) \in c^* \Leftrightarrow \exists x^{\mathbb{R}} [r_n < x < r_m \wedge x \notin c],$$

exists as a consequence of $\text{Proj}^{\mathbb{R}}_{\mathbb{N}}$, hence of Proj_1. It follows that $(r_n, r_m) \subseteq c \Leftrightarrow (n,m) \in (-c^*)$. Finally $c \equiv \bigcup (r_n, r_m) [(n,m) \in (-c^*)]$.

It follows immediately that the Heine-Borel theorem holds for $\mathbb{R}$, i.e. any full open covering of $[a,b]$ reduces to a finite subcovering - since we already know this for countable open coverings.

Stronger consequences are obtainable if we add a uniform version of Proj_1, namely the following axiom for a constant $\exists^{\mathbb{N} \rightarrow \mathbb{N}}_{\mathbb{N}}$:

(i) $(\exists_{\mathbb{N}}^{\mathbb{N} \dot{\rightarrow} \mathbb{N}}) \in \mathbb{S}(\mathbb{N} \times (\mathbb{N} \dot{\rightarrow} \mathbb{N})) \dot{\rightarrow} \mathbb{S}(\mathbb{N})$

(ii) $b \in \mathbb{S}(\mathbb{N} \times (\mathbb{N} \dot{\rightarrow} \mathbb{N})) \wedge a = \exists_{\mathbb{N}}^{\mathbb{N} \dot{\rightarrow} \mathbb{N}}(b) \Rightarrow$

$$\forall n[n \in a \Leftrightarrow \exists g^{\mathbb{N} \dot{\rightarrow} \mathbb{N}} (n,y) \in b].$$

With this uniform version of Proj_1 we derive a union operation on sequences $b = \langle b_y \rangle_{y \in \mathbb{N} \dot{\rightarrow} \mathbb{N}}$ of subsets of $\mathbb{N}$: $n \in \exists_{\mathbb{N}}^{\mathbb{N} \dot{\rightarrow} \mathbb{N}}(b) \Leftrightarrow \exists y^{\mathbb{N} \dot{\rightarrow} \mathbb{N}}(n \in b_y)$. This can be used to obtain a function $\exists{\downarrow}^{\mathbb{N}}$ which decides whether there are descending sequences in $\mathbb{N}$-trees. Given b, to tell whether $\exists y^{\mathbb{N} \dot{\rightarrow} \mathbb{N}} \forall n[\bar{y}(n) \in b]$ let

$$b_y = \begin{cases} \mathbb{N} & \text{if } \forall n[\bar{y}(n) \in b] \\ \Lambda & \text{otherwise} \end{cases}.$$

Thus $\forall n[\bar{y}(n) \in b] \Leftrightarrow 0 \in b_y$ and $\exists y^{\mathbb{N} \dot{\rightarrow} \mathbb{N}} \forall n[\bar{y}(n) \in b] \Leftrightarrow 0 \in \bigcup b_y[y \in \mathbb{N} \dot{\rightarrow} \mathbb{N}]$.

From the axiom for $\exists_{\mathbb{N}}^{\mathbb{N} \dot{\rightarrow} \mathbb{N}}$ also follows corresponding axioms for constants $\exists_{\mathbb{N}}^{\mathbb{R}}$ and $\exists_{\mathbb{N}}^{\mathbb{S}(\mathbb{N})}$. However, even these seem insufficient to derive existence of the outer-measure function $\mu^*(c)$ for subsets c of $\mathbb{R}$. Classically, this is defined as $\inf \mu(G)[G$ is open $\wedge\ c \subseteq G]$. Here we can take it as $\inf \mu(g)[g \in \mathbb{S}(\mathbb{N}) \wedge c \subseteq \bigcup_{n \in g} (r_{n_0}, r_{n_1})]$, where we write $\mu(g)$ for $\mu(\bigcup(g))$. Thus

$$r < \mu^*(c) \Leftrightarrow \exists g^{\mathbb{S}(\mathbb{N})}\{\forall x^{\mathbb{R}}[x \in c \Rightarrow \exists n \in g(r_{n_0} < x < r_{n_1})] \wedge r < \mu(g)\}.$$

This takes the form $r < \mu^*(c) \Leftrightarrow \exists g^{\mathbb{S}(\mathbb{N})}\{\forall x^{\mathbb{R}}(x,g) \in c' \wedge r < \mu(g)\}$; the inner quantifier $\forall x\,\mathbb{R}(\ldots)$ can be applied using $\exists_{\mathbb{S}(\mathbb{N})}^{\mathbb{R}}$ but not with $\exists_{\mathbb{N}}^{\mathbb{R}}$.