# COHOMOLOGICAL DESCENT

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Contents



## Introduction

In classical Čech theory, we "compute" (or better: filter) the cohomology of a sheaf when given an open covering. Namely, if $X$ is a topological space, $\mathfrak{U} = \{U_i\}$ is an indexed open covering, and $\mathscr{F}$ is an abelian sheaf on $X$, then we get a Čech to derived functor spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathfrak{U}, \underline{\mathrm{H}}^q(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F}),$$

where $\underline{\mathrm{H}}^q(\mathscr{F})$ is the presheaf whose value on an open $U$ is $\mathrm{H}^q(U, \mathscr{F}|_U)$ (and we use the contravariant pullback functoriality). In particular, $\underline{\mathrm{H}}^0(\mathscr{F}) = \mathscr{F}$ and $\underline{\mathrm{H}}^q(\mathscr{F})$ sheafifies to be zero if $q > 0$. Of course, if $\mathscr{F}$ has vanishing cohomology on the finite overlaps of the $U_i$'s then this degenerates to give an edge isomorphism

$$\mathrm{H}^n(\mathfrak{U}, \mathscr{F}) \simeq \mathrm{H}^n(X, \mathscr{F}).$$

The Čech to derived functor spectral sequence is also natural in the space $X$ and the open cover $\mathfrak{U}$. Verdier's theory of hypercoverings somewhat generalized the scope of these techniques, but still remained within the framework of using "covers" relative to some Grothendieck topology. A rather dramatic improvement was given by Deligne in his theory of cohomological descent. This theory is a fantastic derived category generalization of Grothendieck's descent theory for sheaves (see Lemma 6.8 and the discussion preceding it for the precise connection).

As one application, for a *smooth* projective variety over $\mathbf{C}$, there is a nice theory of Hodge structures on the topological cohomology of $X$ (by which we really mean the cohomology of $X(\mathbf{C})$). In [D], Deligne generalizes this to a theory of mixed Hodge structures with no smoothness conditions. However, ultimately his construction rests on the amazing possibility of being able to systematically use iterated applications of resolution of singularities to "compute" the topological cohomology of an arbitrary projective variety $X_{/\mathbf{C}}$ in terms of the topological cohomology of smooth (projective) $\mathbf{C}$-schemes which are *proper* over $X$ (and highly disconnected!). In a nutshell, Deligne developed a way to use Čech-like methods to compute cohomology relative to a topology by means of certain maps $X_p \to X$ which need *not* be even remotely like covering maps for the given topology. Of course, working with such "out of the topology" covers requires some pretty strong conditions to hold, but such conditions are satisfied in the case of proper maps (due to the proper base change theorem).

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There are two essential ingredients for making cohomological descent work: the simplicial theory of hypercoverings, which vastly generalizes the use of Čech theory to "compute" (via a spectral sequence) the cohomology of a sheaf via a covering of the space, and derived category techniques (such as adjointness of $\mathbf{R}f_*$ and $f^*$ for a continuous map $f : X \to Y$ between topological spaces) which are needed to even formulate the *definition* of cohomological descent (let alone to prove it is a flexible notion).

Although one aim of the theory to construct a (functorial!) spectral sequence of *down-to-earth* objects

$$E_1^{p,q} = \mathrm{H}^q(X_p(\mathbf{C}), \underline{A}) \Rightarrow \mathrm{H}^{p+q}(X(\mathbf{C}), \underline{A})$$

for any abelian group $A$ and suitable auxiliary smooth projective (usually disconnected!) $\mathbf{C}$-schemes $X_p$ constructed cleverly by resolution of singularities (see Theorem 4.16, (6.3), and Theorem 7.9), the use of *derived category* methods in the construction of such a spectral sequence seems to be unavoidable. The simplicial theory of hypercovers provides a *single* framework which subsumes both "exotic" examples such as the one above with $X_p \to X$ that are proper, as well as the spectral sequences coming from classical Čech theory. The astute reader will note that the Čech spectral sequence is at the $E_2$ stage, while the one we just wrote down (ambiguously) in a "proper hypercover" case was at the $E_1$ stage. As we will see later, it is an $E_1$ term that one always gets for free, and in the Čech case one can actually explicate the next step very concretely and thereby get the expected $E_2$-terms (and of course, one really gets the entire classical spectral sequence on the nose, not just $E_2$-term objects).

After some initial motivating examples from topology are discussed in §1, for conceptual clarity (as well as generality) we will discuss the theory of simplicial objects in any category in §2. We place particular emphasis on the all-important coskeleton functor in §3. Then we will see in §4 how, for suitable categories (such as schemes, topological spaces, or any Grothendieck site), this gives rise to the notion of hypercoverings (generalizing ordinary coverings as used in Čech theory). We will see how resolution of singularities gives rise to particularly nice proper hypercoverings of any separated scheme of finite type over a field. In §5 we discuss some basic aspects of the theory of simplicial homotopy, with special focus on how it interacts with the coskeleton functors, as this is rather important for Deligne's main results in the theory.

Once these simplicial foundations have been explained, we will set up the basic formalism of cohomological descent in §6 and see why it is a derived category version of classical descent theory for sheaves (hence explaining the name). The deepest part (in terms of non-formal input), as well as (for me) the most interesting part, is to prove that there are ways to construct interesting examples of cohomological descent, such as proper surjective hypercoverings. It is in *establishing* the cohomological descent property of proper hypercoverings (both in the topological category as well as in the étale topology) that we will have to use non-formal input – the proper base change theorem – and inductive simplicial techniques in terms of coskeleta. This issue is treated in §7. Theorems 7.5, 7.10, and 7.22 are really the fundamental results in the theory, and all of their proofs rest on the use of bisimplicial methods (and extreme cleverness).

For someone interested in [D], here is where you can find proofs in these notes for the facts from the theory of cohomological descent which Deligne states in [D] (with generic reference to [SGA4, Exp Vbis] for proofs):

- [D, 5.3.5(I)] is Theorem 7.5 (more general: see Theorem 7.9).
- [D, 5.3.5(II)] is Theorem 7.7.
- [D, 5.3.5(III)] is Theorem 7.2.
- [D, 5.3.5(IV)] is Theorem 7.15 (with the help of Corollary 3.11).
- [D, 5.3.5(V)] is Theorem 7.22 (actually, this is not discussed in [SGA4], but follows from an extension of preceding methods).

Also see Example 7.8 for an application of cohomological descent for proper hypercoverings in the context of computing the cohomology of a space in terms of a locally finite covering by closed sets.

The theory of cohomological descent has applications far beyond the construction of mixed Hodge structures. For example, de Jong's resolution theorem [dJ] makes it possible – via cohomological descent – to prove quasi-unipotence of inertia actions in the strongest possible form (with an "independence of $\ell$" aspect) for the $\ell$-adic cohomology of an arbitrary separated scheme of finite type over the fraction field of a discrete valuation ring. Earlier results of Grothendieck in this direction were only valid for a residue field with non-trivial $\ell$-adic cyclotomic character (e.g., algebraically closed residue field wasn't included) and more significantly did not give results which were independent of $\ell$. Moreover, it should be noted that de Jong's theorem is adequate for Deligne's hypercovering methods in [D], so the dependence on Hironaka in [D] is eliminated by means of [dJ]. In Theorem 4.16 we review the role of resolution of singularities in the construction of regular proper hypercoverings.

When I first wanted to learn the theory of cohomological descent years ago, I tried to read the discussion in [SGA4], but the intense amount of topos theory in that discussion obscured (for me at that time) what was going on. There were also other references which tended to use a big dosage of homotopy theory (about which I knew nothing), so it all seemed rather remote. But once I finally got over my psychological crutches and learned a bit of topos theory and homotopical category theory, I was able to understand what was going on in the cohomological descent discussion in [SGA4] and to my pleasant surprise found that the topos theory was (mostly) a red herring and one could develop the central ideas of the theory in the generality with which it is frequently used (such as in [D], or for schemes with the étale topology, or for other similar Grothendieck topologies) without really requiring any fancy general topos theory at all, though of course sacrificing some super-generality in the process. I decided to write up these notes explaining Deligne's theory with the hope that they would enable more people who are comfortable with derived categories and are as ignorant as I was about homotopy theory to become familiar with the beautiful theory of cohomological descent without mistakenly thinking that first they have to learn a lot of topos theory and advanced homotopy theory to understand what's going on. Of course, these notes should also make it easier for the interested reader to study the wealth of additional ideas in [SGA4, Exp Vbis] which we don't address here.

The informed reader will readily check that everything we do also applies pretty much verbatim, with occasional minor modification, to other interesting sites (such as non-archimedean analytic spaces with the Tate topos, or schemes with quasi-coherent sheaves and quasi-compact quasi-separated morphisms and the fppf topology, or pretty much any reasonable ringed topos for which pullback functors on sheaves are *exact*).

A caveat. In these notes, we take a partly ad hoc approach to the theory of multisimplicial objects. In a couple of places, bisimplicial objects play an essential role in proofs and we have developed what we need to make those arguments work. That said, the bisimplicial ideas in the proofs of Theorem 7.5(1) and Theorem 7.17 are not presented in the slick manner of [SGA4] for the simple reason that I couldn't fully understand Deligne's arguments at those "bisimplicial" steps and hence came up with alternative arguments. I suspect these alternative arguments ultimately boil down to explications of Deligne's slicker point of view, but the reader will see that our arguments at these steps are rather long. If any reader of these notes can understand Deligne's more efficient ways of dealing with these particular proofs, please contact me!

Conventions. Following (what should be) standard conventions, chain complexes in an abelian category have differentials which increase degrees and cochain complexes in an abelian category have differentials which decrease degrees. Passing to the opposite category interchanges these notions. Also, if $C$ is a category and $S$ is an object, we define the *slice category* $C/S$ to be the category whose objects are morphisms $X \to S$ (with evident notion of morphism between two such objects).

Although at the beginning of these notes we treat topological spaces and schemes as separate entities (as the notions of fiber product and properness in the two categories are not compatible with the functor that assigns to each scheme its underlying topological space), after a certain amount of time we just adopt the terminology "space" for an object in either the category of topological spaces or the category of schemes (with the étale topology), and we'll say a map in the topological category is "étale" if it is a local isomorphism. This enables us to be a bit more efficient with the exposition and creates no risk of confusion (and any site which is similar to these sites would work just as well). We could instead have opted to use the more uniform and universal language of ringed topoi so as to handle all examples at once, and of course the whole point of topoi is to put such arguments into a single universally applicable framework. However, to have written in such style would have defeated the expository purpose of these notes (as those who prefer topoi would probably just read the exposition in [SGA4] anyway).

## 1. Motivation for simplicial methods

Let $X$ be a topological space. The proofs of the basic results in Čech theory are very combinatorial (mostly index-chasing), and we wish to find a suitably abstract context for those techniques. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an indexed open covering of $X$, and $U = \coprod U_i$ the disjoint union of the $U_i$'s. We emphasize that we do *not* assume our index set to be ordered, and we recall that when computing with Čech theory, it (functorially) does not matter whether one universally computes with ordered covers or unordered covers. Of course, even a finite (non-empty) covering gives rise to an infinitely long Čech complex when using unordered covers (e.g., one has an $(n+1)$-fold "self-overlap" $U_i$ of each $U_i$ with itself for each tuple $(i, i, \dots, i) \in I^{n+1}$ for each $n \geq 0$). We will see that for theoretical purposes it is the unordered case which fits best into the abstract simplicial formalism we wish to set up.

Define $X_0 = U = \coprod U_i$, $X_1 = U \times_X U$, and in general define $X_n = U^{\times(n+1)}$ to be the $(n+1)$-fold fiber product of $U$ over $X$. Note that

$$X_1 = \coprod_{(i,j) \in I^2} (U_i \cap U_j), \quad X_2 = \coprod_{(i,j,k) \in I^3} (U_i \cap U_j \cap U_k),$$

and so on. The indexing for $X_n$ is by $I^{n+1}$, so there is no ruling out of repeated coordinates. For example, the overlap $U_i \cap U_j = U_j \cap U_i$ really shows up *twice* in $X_1$, for the pair $(i,j)$ and the pair $(j,i)$, unless of course $i = j$, in which case this $U_i$ term shows up once (corresponding to $(i,i) \in I^2$). This is in contrast with what one encounters in Čech theory for ordered open covers. For $i \neq j$ the first projection $X_1 \to X_0 = U$ sends the $(i,j)$ copy of $U_i \cap U_j$ to $U_i$ via the canonical map and sends the $(j,i)$ copy of this overlap to $U_j$ via the canonical map. Note that the diagonal section

$$\Delta : X_0 = U \to U \times_X U = X_1$$

sends each $U_i$ to the $(i,i)$-copy of $U_i$ in $X_1$ via the identity map.

Let us label the $n+1$ factors $U$ of the fiber product $X_n$ with the elements of the ordered set

$$[n] := \{0, \dots, n\}$$

(in particular, we view these fiber powers $X_n$ as having factors $U$ with a specified ordering). For each $n \geq 1$ and $0 \leq j \leq n$ there are natural projection maps

$$p_n^j : X_n \to X_{n-1}$$

away from the $j$th factor, described by

$$(u_0, \dots, u_n) \mapsto (u_0, \dots, \widehat{u_j}, \dots, u_n).$$

Likewise, for $n \geq 0$ and $0 \leq j \leq n$ there are inclusion maps

$$\iota_n^j : X_n \to X_{n+1}$$

which "repeat" the $j$th factor twice:

$$(u_0, \dots, u_n) \mapsto (u_0, \dots, u_j, u_j, u_{j+1}, \dots, u_n)$$

(this is just obtained by inserting the diagonal map of $U$ on the $j$th slot of the fiber product $X_n$).

As an example, $\iota_0^0 : X_0 \to X_1$ is just the diagonal map for $U$, and so is a section to both $p_1^0$ and $p_1^1$, while $\iota_1^0 : X_1 \to X_2$ is given by $(u_0, u_1) \mapsto (u_0, u_0, u_1)$ and hence is a section to both $p_2^0$ and $p_2^1$, but not to $p_2^2$. Similarly,

$$\iota_1^1 : (u_0, u_1) \mapsto (u_0, u_1, u_1)$$

is a section to $p_2^1$ and $p_2^2$ but not to $p_2^0$. In general, by staring at pictures such as (2.3) and walking along the tree of arrows, one readily checks the following identities (to be generalized vastly later on):

**Lemma 1.1.** *For $0 \leq j < j' \leq n+1$ we have*

$$p_n^j \circ p_{n+1}^{j'} = p_n^{j'-1} \circ p_{n+1}^j.$$

*For $0 \leq j \leq j' \leq n$ we have*

$$\iota_{n+1}^j \circ \iota_n^{j'} = \iota_{n+1}^{j'+1} \circ \iota_n^j.$$

*Finally, we have*

$$p_n^j \circ \iota_{n-1}^{j'} = \begin{cases} \iota_{n-2}^{j'-1} \circ p_{n-1}^j, & \text{if } 0 \le j < j' \le n-1 \\ \text{id}, & \text{if } 0 \le j' \le j \le j'+1 \le n \\ \iota_{n-2}^{j'} p_{n-1}^{j-1}, & \text{if } 0 \le j'+1 < j \le n \end{cases}$$

As an example, for $n \ge 1$, the "identity" composites in Lemma 1.1 reflect the fact that $\iota_{n-1}^{j'}$ is a section to $p_n^j$ for $j = j', j'+1$.

If we consider the "covering" map

$$\varepsilon : X_0 = U = \coprod U_i \to X$$

(for which there is no natural section!), then observe that for a fixed $n \ge 1$ all composites

$$\varepsilon \circ p_1^{j_1} \circ \cdots \circ p_{n-1}^{j_{n-1}} \circ p_n^{j_n} : X_n \to X \tag{1.1}$$

coincide (and are just the structure map for the fiber powers of $U$ over $X$). If we denote this map $p_n : X_n \to X$ then all $X_n$'s become spaces "over" $X$ and one readily checks:

**Lemma 1.2.** *All maps $\iota_n^j$, $p_n^j$ are morphisms over $X$, with $p_n \circ p_{n+1}^j = p_{n+1}$ for all $0 \le j \le n+1$.*

The entire structure of Čech theory for the covering $\mathfrak{U} = \{U_i\}$ can be recovered from the data of the maps $p_n^j$, $\iota_n^j$, and $\varepsilon$ (which we might call $p_0 = p_0^0$). Let's see how this goes. For an abelian sheaf $\mathscr{F}$ on $X$, define $\mathscr{F}^n = p_n^*\mathscr{F}$ on $X_n$ for $n \ge 0$ (so $\mathscr{F}^0 = \varepsilon^*\mathscr{F}$). In view of the concrete description of $X_n$ in terms of a disjoint union of $(n+1)$-fold overlaps of the $U_i$'s, we see that $\mathscr{F}^n$ just encodes the restrictions of $\mathscr{F}$ to all of these overlaps. Moreover, the abelian sheaf

$$\widetilde{\mathscr{F}}^n := p_{n*}p_n^*\mathscr{F} = p_{n*}\mathscr{F}^n$$

is just the $n$th term $\mathscr{C}^n(\mathfrak{U}, \mathscr{F})$ of the Čech complex for $\mathscr{F}$ with respect to the open covering $\mathfrak{U} = \{U_i\}$ (again, we remind the reader that we compute Čech theory using unordered index sets). Since $p_n \circ p_{n+1}^j = p_{n+1}$, we get a natural map

$$\delta_j^n : \widetilde{\mathscr{F}}^n = p_{n*}p_n^*\mathscr{F} \to p_{n*}p_{n+1}^j{}_* p_{n+1}^j{}^* p_n^*\mathscr{F} \simeq p_{n+1}{}_* p_{n+1}^*\mathscr{F} = \widetilde{\mathscr{F}}^{n+1}$$

and the map

$$\partial^n = \sum_{j=0}^{n+1} (-1)^j \delta_j^n : \widetilde{\mathscr{F}}^n \to \widetilde{\mathscr{F}}^{n+1} \tag{1.2}$$

is *exactly* the natural differential in degree $n$ in the Čech complex $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$. The fact that $\partial^{n+1} \circ \partial^n = 0$ for $n \ge 0$ can be deduced purely formally from the relations on the $p_n^j$'s in Lemma 1.1. Moreover, the natural map

$$\partial^{-1} : \mathscr{F} \to \varepsilon_*\varepsilon^*\mathscr{F} = \widetilde{\mathscr{F}}^0$$

is exactly the canonical augmentation in degree 0 (and by Lemma 1.2 we see formally that $\partial^0 \circ \partial^{-1} = 0$). Thus, we see that the entire structure of Čech theory can be obtained from the maps $p, \iota, \varepsilon$ with the help of canonical adjointness maps, without needing to explicitly refer to the cartesian power aspect of the construction that gave rise to these maps in the first place.

Now an important point arises: in the Čech theory, we didn't really use the maps $\iota$ very much, and we effectively collapsed all the $p_n^j$'s for fixed $n$ into the single map $\partial^n$ by means of the additive structure on abelian sheaves. The crucial point of simplicial theory is to keep track of *all* the maps $p_n^j$, $\iota_n^j$, $\varepsilon$: such data is *much* more fundamental than the cartesian power specificity, for (as we shall see) there are ways to "refine" the $X_n$'s such that the cartesian power structure is lost but the information of maps satisfying relations as in Lemma 1.1 is *not* lost. Such techniques of "refinement" will make sense within the framework of hypercoverings.

Let us summarize by indicating the two biggest drawbacks in Čech theory (from the point of view of the more general theory of hypercoverings, to be discussed later):

- The single map $\varepsilon : X_0 \to X$ determines everything else (via fiber powers over $X$, etc.), and refining an open covering only gives rise to "refinements coming from degree 0". We want to be able to modify things in higher degrees without affecting lower degrees. The ordinary theory of coverings is too restrictive for carrying this out.
- The Čech complex only uses the $p_n^j$'s, not the $\iota_n^j$'s, and for fixed $n$ doesn't even directly keep track of all of the $p_n^j$'s separately (rather, only the alternating sum $\partial^n$ is encoded).

We will now consider one further example from topology in which the preceding structure is visible but which again suffers from a similar drawback of not "keeping track of all the data" in classical applications. Once again let $X$ be a topological space, and let $\Delta_n(X)$ denote the *set* of $n$-simplices in $X$. That is, $\Delta_n(X)$ is the set of all continuous maps $\varphi : \Delta[n]_{\mathbf{R}} \to X$ where $\Delta[n]_{\mathbf{R}} \subseteq \mathbf{R}^{n+1}$ is the standard $n$-simplex

$$\Delta[n]_{\mathbf{R}} = \{(t_0, \dots, t_n) \in \mathbf{R}^{n+1} \,|\, 0 \le t_j \le 1,\ \sum t_j = 1\}.$$

For $n \ge 1$ and $0 \le j \le n$ there are natural "face" maps

$$D_n^j : \Delta_n(X) \to \Delta_{n-1}(X)$$

given by composing an $n$-simplex $\varphi : \Delta[n]_{\mathbf{R}} \to X$ with the (continuous) inclusion

$$\partial_n^j : \Delta[n-1]_{\mathbf{R}} \to \Delta[n]_{\mathbf{R}}$$

onto the $j$th "face" via

$$(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1})$$

(so this map $D_n^j$ assigns to each $n$-simplex $\varphi$ its $j$th face $\varphi \circ \partial_n^j$, an $(n-1)$-simplex). For $n \ge 0$ and $0 \le j \le n$, we also have "degeneracy" maps

$$s_n^j : \Delta_n(X) \to \Delta_{n+1}(X)$$

given by composing each $n$-simplex $\varphi : \Delta[n]_{\mathbf{R}} \to X$ with the map $\Delta[n+1]_{\mathbf{R}} \to \Delta[n]_{\mathbf{R}}$ defined by

$$(t_0, \dots, t_{n+1}) \mapsto (t_0, \dots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \dots, t_{n+1}).$$

This converts each $n$-simplex into a "degenerate" $(n+1)$-simplex.

By essentially the same index-pushing as in the preceding Čech situation, one checks that the relations in Lemma 1.1 continue to hold if $\partial_n^j$ replaces $p_n^j$ and $s_n^j$ replaces $\iota_n^j$. If we also define

$$\Delta_{-1}(X) = \mathrm{Hom}_{\mathrm{Top}}(\emptyset, X) = \{\emptyset\}$$

to be the singleton set, there is a unique (!) map $D_0^0 = \varepsilon : \Delta_0(X) \to \Delta_{-1}(X)$ for which the analogues of Lemma 1.2 continues to hold.

Defining $S_n(X)$ to be the free abelian group generated by the sets $\Delta_n(X)$ and

$$S^n(X) = \mathrm{Hom}_{\mathrm{Ab}}(S_n(X), \mathbf{Z}) = \mathrm{Hom}_{\mathrm{Set}}(\Delta_n(X), \mathbf{Z})$$

for $n \ge -1$, we can define natural maps

$$\partial_n = \sum_{j=0}^{n} (-1)^j D_n^j : S_n(X) \to S_{n-1}(X)$$

for $n \ge 0$ and

$$\mathrm{d}^n = \sum_{j=0}^{n+1} (-1)^j D_{n+1}^j : S^n(X) \to S^{n+1}(X)$$

for $n \ge -1$. The analogue of Lemmas 1.1 and 1.2 again ensure that these define complexes of abelian groups, and these are just the usual (augmented) simplicial chain and cochain complexes from algebraic topology.

We stress that the simplicial (co)chain complex of $X$ is *less* information than the data of the individual *set* maps $D_n^j$ and $s_n^j$. A key philosophical point in simplicial topology is to retain all of this data and not to just ignore the $s_n^j$'s and collapse the $D_n^j$'s (for fixed $n$) into an alternating sum on abelian group objects.

## 2. Simplicial Objects

The two examples considered in §1 seem to be formally quite similar in terms of the structure they encode. This is made precise by means of the theory of simplicial objects in a category. We begin with a basic definition.

**Definition 2.1.** For an integer $n \geq -1$, let $[n] = \{0, \ldots, n\}$ be an ordered set in the usual manner (so $[-1] = \emptyset$). We define $\Delta^+$ to be the category of these objects, with $\mathrm{Hom}_{\Delta^+}([n],[m])$ denoting the set of non-decreasing (i.e., monotonically increasing) maps of ordered sets $[n] \to [m]$. We denote by $\Delta$ the full subcategory of objects $[n]$ with $n \geq 0$.

Note that $\Delta^+$ has an initial object $[-1]$ with no object other than $[-1]$ having any morphism to $[-1]$, whereas $\Delta$ does not have an initial object (e.g., there exist two distinct maps $[0] \to [1]$). This fact is the abstraction corresponding to the distinction between classical chain complexes in non-negative degrees and augmented chain complexes. A typical map $\phi : [n] \to [m]$ in $\Delta$ amounts to collapsing several strings of adjacent elements in $[n]$ (if $\phi$ isn't injective) and then sticking in some gaps (if $\phi$ isn't surjective). The basic examples of morphisms are

$$\partial_j^n : [n-1] \to [n], \ \ \sigma_j^n : [n+1] \to [n] \tag{2.1}$$

for $n \geq 0$ and $0 \leq j \leq n$, with $\partial_j^n$ the unique increasing injection whose image does not contain $j \in [n]$, and $\sigma_j^n$ the unique increasing surjection which hits $j \in [n]$ twice. These maps embody the ultimate abstraction of the index-chasing in classical Čech theory, and it is by thinking in terms of the categories $\Delta$ and $\Delta^+$ that will be be able to formulate a vast generalization of Čech theory which allows us to do calculations in a much wider range of situations.

By thinking in terms of how a map $\phi : [n] \to [m]$ in $\Delta$ collapses adjacent integers or inserts gaps in the target, one readily checks that $\phi$ can be *uniquely* expressed as a composite

$$\phi = \partial_{j'_s}^m \circ \partial_{j'_{s-1}}^{m-1} \circ \cdots \circ \partial_{j'_1}^{m-s+1} \circ \sigma_{j_r}^{n-r} \circ \cdots \circ \sigma_{j_1}^{n-1} \tag{2.2}$$

with $0 \leq j_r < \cdots < j_1 < n$, $0 \leq j'_1 < \cdots < j'_s \leq m$, and $m - s = n - r$ (necessarily equal to the size of the image of $\phi$). Of course, we understand the empty expression on the right (i.e., $r = s = 0$, so $m = n$) to denote the identity map on $[m]$.

The maps $\partial_j^n$ and $\sigma_j^n$ satisfy relations analogous to those in Lemma 1.1, and although there are plenty of other relations among the maps in $\Delta$, it is an interesting and crucial fact that *all* relations among morphisms in $\Delta$ can be derived from the ones analogous to the relations in Lemma 1.1. This is given by:

**Lemma 2.2.** *The category $\Delta$ on the objects $[n]$ for $n \geq 0$ is generated by identity maps and the morphisms* (2.1) *for $0 \leq j \leq n$, with the relations of associativity and*

$$\partial_j^{n+1}\partial_{j'}^n = \partial_{j'}^{n+1}\partial_{j-1}^n$$

*for $0 \leq j' < j \leq n+1$,*

$$\sigma_j^n\sigma_{j'}^{n+1} = \sigma_{j'}^n\sigma_{j+1}^{n+1}$$

*for $0 \leq j' \leq j \leq n$, and*

$$\sigma_j^{n-1}\partial_{j'}^n = \begin{cases} \partial_{j'}^{n-1}\sigma_{j-1}^{n-2}, & \text{if } 0 \leq j' < j \leq n-1 \\ \mathrm{id}_{[n-1]}, & \text{if } 0 \leq j \leq j' \leq j+1 \leq n \\ \partial_{j'-1}^{n-1}\sigma_j^{n-2}, & \text{if } 0 \leq j+1 < j' \leq n \end{cases}$$

*Proof.* By (2.2), the asserted list of maps does generate all maps in $\Delta$. We just have to check that all relations among morphisms are in fact obtained from the ones listed in the statement of the lemma (together with the associativity of composition). This is shown in [GZ, p. 24].

■

Observe that the data $(X_n, p_n^j, \iota_n^j)_{n\geq 0}$ and $(\Delta_n(X), \partial_n^j, s_n^j)_{n\geq 0}$ from §1 are really just *contra*variant functors from the category $\Delta$ to the categories of topological spaces and sets respectively. That is, to each object $[n]$ in $\Delta$ we associate the object $X_n$ (resp. $\Delta_n(X)$) and to each map $\phi : [n] \to [m]$ in $\Delta$ we assign a map $X(\phi) : X_m \to X_n$ (resp. $\Delta(\phi) : \Delta_m(X) \to \Delta_n(X)$) defined as follows. If $\phi = \partial_j^n : [n-1] \to [n]$ then $X(\phi) := p_n^j$ and $\Delta(\phi) := D_n^j$. If $\phi = \sigma_j^n$, then $X(\phi) := \iota_n^j$ and $\Delta(\phi) := s_n^j$. Notice that the relations in Lemma 2.2 are inherited under these constructions, as given by Lemma 1.1 and its analogue for the classical simplex construction. This is crucial, because what we would like to do for general $\phi$ is use the unique factorization (2.2) to define $X(\phi)$ and $\Delta(\phi)$ in the unique possible manner compatible with associativity, contravariance, and the previous definitions made in the special cases $\phi = \partial_j^n, \sigma_j^n$. However, the factorization (2.2) is poorly behaved with respect to composition, so in principle we're faced with a rather painful problem of chasing relations to ensure well-definedness, compatibility with composition, etc.

This is where Lemma 2.2 saves the day: since we have Lemma 1.1 and its analogue for the simplex construction, by Lemma 2.2 it is legitimate to define $X(\phi)$ in accordance with the desired recipe resting on (2.2). Hence, the two examples in §1 really are genuinely contravariant functors from $\Delta$ to the categories of topological spaces and sets respectively. This interpretation of the §1 examples automatically includes all of the maps we have been studying, as well as all of the relations which we have considered among these maps.

With this connection between the examples in §1 and the abstract category $\Delta$ now understood, we can give the ultimate abstraction of the examples from §1. We emphasize that the specificity of cartesian powers and topological simplices is completely eliminated in the following definition, and all we retain is the essential structure of many *maps*:

**Definition 2.3.** Let $C$ be a category. A *simplicial object* (or *simplicial complex*) in $C$ is a contravariant functor $X_\bullet : \Delta \to C$. That is (by Lemma 2.2!), it is a collection of objects $X_n$ in $C$ for all $n \geq 0$ and (face and degeneracy) maps

$$\mathrm{d}_n^j : X_n \to X_{n-1}, \ \ s_n^j : X_n \to X_{n+1}$$

for $0 \leq j \leq n$ satisfying the *opposite* of the relations in Lemma 2.2.

These objects form a category $\mathrm{Simp}(C)$ with morphisms $X'_\bullet \to X_\bullet$ just natural transformations (i.e., collections of morphisms $X'_n \to X_n$ which commute with the degeneracy and face maps on both sides).

*Example* 2.4. If $C$ is a category with products and $Y_0$ is an object in $C$, then an example of a simplicial object in $C$ is provided by the cartesian powers of $Y_0$ with the evident degeneracy and face maps as in our earlier simplicial formulation of Čech theory. This is typically drawn as a diagram:

$$Y_0 \underset{\mathrm{d}}{\overset{s}{\rightleftarrows}} Y_0 \times Y_0 \underset{\mathrm{d}}{\overset{s}{\rightleftarrows}} Y_0 \times Y_0 \times Y_0 \underset{\mathrm{d}}{\overset{s}{\rightleftarrows}} Y_0 \times Y_0 \times Y_0 \times Y_0 \ldots \tag{2.3}$$

where, for example, the two maps $s_1^0, s_1^1 : Y_0 \times Y_0 \to Y_0 \times Y_0 \times Y_0$ are given by

$$s_1^0(y, y') = (y, y, y'), \ \ s_1^1(y, y') = (y, y', y').$$

Note that we *never* "switch" the order of coordinates.

A general simplicial object in any category $C$ is also essentially a diagram of arrows as in (2.3) in which the terms need *not* be cartesian powers of the term in degree 0 but where we require that the arrows do satisfy the evident *relations* which are satisfied by the maps in (2.3) (i.e., the opposite of the relations in Lemma 2.2).

Intuitively, we'd like to imagine a simplicial object $X_\bullet$ as a generalized CW-complex, with $n$-cells given by $X_n$ and functoriality giving the gluing data. In fact, there is a "geometric realization" functor from simplicial objects in a category to topological spaces, but we will not make any use of this. Nevertheless, the intuitive picture provided by thinking about $X_\bullet$ as a CW-complex is very suggestive and helpful.

There is an evident extension of the preceding definition to the case of the category $\Delta^+$ containing the initial object $[-1] = \emptyset$, and we call the corresponding notion an *augmented simplicial object* in $C$. By thinking about how the category $\Delta^+$ is made from the category $\Delta$ by "formally adjoining" an initial object $[-1]$ (check!), it is straightfoward to check that to give an augmented simplicial object in $C$ is to give a simplicial object $X_\bullet$ and another object $X_{-1}$ of $C$ equipped with a map $\partial_0^0 : X_0 \to X_{-1}$ (to be $X_\bullet(\varepsilon)$ with $\varepsilon : [-1] \to [0]$ the unique map) such that for any $n \geq 0$, all composites

$$\partial_0^0 \circ \partial_1^{j_1} \circ \cdots \circ \partial_n^{j_n} : X_n \to X_{-1}$$

coincide (cf. (1.1)). This also then gives each $X_n$ a structure of "object over $X_{-1}$" and then all maps in $X_\bullet$ are over $X_{-1}$. In other words, to give an augmented simplicial object in $C$ with a *specified* augmentation object $X_{-1}$ is nothing more or less than a simplicial object in the "slice category" $C_{/X_{-1}}$ of objects in $C$ over $X_{-1}$.

**Definition 2.5.** We write $\mathrm{Simp}^+(C)$ to denote the category of augmented simplicial objects in $C$.

We will usually write $a : X_\bullet \to S$ to denote an augmented simplicial object with $S$ in degree $-1$ and $X_\bullet$ the part in degrees $\geq 0$. There are then canonical maps $a_n : X_n \to S$ for all $n \geq 0$. We stress that $a$ isn't really a "morphism", but is more shorthand notation for a collection of data. In other cases we may write $X_\bullet$ to denote the entire augmented structure (rather than just the "simplicial part" in degrees $\geq 0$). The context should always make clear what we mean, but sometimes we will also write $X_\bullet/S$ to denote an augmented simplicial object with $S$ in degree $-1$ and $X_\bullet$ the simplicial part.

*Example* 2.6. A somewhat boring example of a simplicial object (but one which will actually useful later on) is to take $X_\bullet$ with $X_n = S$ a fixed object in $C$ for all $n \geq 0$ and to take all simplicial maps among the $X_n$'s to be the identity on $S$. This sort of example is called a *constant* simplicial object. If we fix an augmentation structure by the identity on $S$, we call the result a *constant augmented* simplicial object.

**Definition 2.7.** A convariant functor $X : \Delta \to C$ will be called a *cosimplicial object* (or *cosimplicial complex*) in $C$, with $\mathrm{Cosimp}(C)$ denoting the corresponding category. The augmented variant is defined and explicated in the evident manner.

Note that $\mathrm{Simp}(C)^{\mathrm{opp}} = \mathrm{Cosimp}(C^{\mathrm{opp}})$.

*Example* 2.8. We begin with a non-example by providing a word of warning about a possible example of cosimplicial object that might spring to mind: the data of the sheaves $\mathscr{F}^n = p_n^* \mathscr{F}$ on the $X_n$'s from §1 with maps

$$p_n^{j\,*} \mathscr{F}^n \to \mathscr{F}^{n+1}, \;\; \iota_n^{j\,*} \mathscr{F}^n \to \mathscr{F}^{n-1} \tag{2.4}$$

on $X_{n+1}$ (for $n \geq 0$) and $X_{n-1}$ (for $n \geq 1$) respectively. Recall that these maps were defined via adjunction morphisms for pushfoward/pullback with the help of the identities

$$p_{n+1} = p_n \circ p_n^j, \;\; p_{n+1} \circ \iota_n^j = p_n \circ p_n^j \circ \iota_n^j = p_n.$$

This data generally does *not* constitute a "cosimplicial sheaf" for the midly annoying technical reason that these $\mathscr{F}^n$'s live on different spaces and hence are not really objects of a common category (of sheaves). Formally we'd like to say that the maps in (2.4) should play the respective roles of $\partial_j^{n+1}$ ($n \geq -1$) and $\sigma_j^{n-1}$ ($n \geq 1$) since natural analogues of the relations in Lemma 2.2 do clearly hold, thanks to the "opposite" of the relations in Lemma 1.1 (upon passing to pullbacks). This should not prevent us from thinking about the $\mathscr{F}_n$'s and the data in (2.4) as if it were a "cosimplicial sheaf" or perhaps more accurately a "sheaf on $X_\bullet$". We will later give a precise definition of such concepts (see Definition 6.1 and Example 6.2).

Of course, this sort of example is clearly of fundamental nature, and Deligne's way of generalizing the situation and working with it was to replace the sheaf categories on the various $X_n$'s and adjoint pair functors $(p_{n*}^j, p_n^{j*})$ and $(\iota_{n*}^j, \iota_n^{j*})$ between them with ringed topoi and suitable geometric morphisms between them, and to just develop a general theory in the context of simplicial ringed topoi to solve all problems at once. However, this generality comes at a price: in order to have a derived pullback which is well-posed on "bounded below" derived categories, Deligne needs to pay careful attention to flatness issues. We will only be discussing pullback maps in the level of abelian sheaves (on a topological space, scheme with étale topology, or on any site), for which $f^*$ is always exact. This restriction is one reason we don't need to carry around the amount of technical baggage that accompanies [D]. In any case, do keep in mind that the sheaf example from §1 is, strictly speaking, not quite a cosimplicial object in the sense of Definition 2.3.

*Example* 2.9. Actually, we can make an honest cosimplicial object out of the non-example above by considering the pushfoward sheaves $\widetilde{\mathscr{F}}^n = p_{n*}p_n^*\mathscr{F}$ on the common space $X = X_{-1}$. We use the old face map

$$\partial_j^{n+1} = \mathrm{d}_j^n : \widetilde{\mathscr{F}}^n \to \widetilde{\mathscr{F}}^{n+1}$$

from (1.2) and we define the degeneracy maps

$$\sigma_j^n : \widetilde{\mathscr{F}}^{n+1} = p_{n+1\,*}p_{n+1}^*\mathscr{F} \to p_{n*}p_n^*\mathscr{F} = \widetilde{\mathscr{F}}^n$$

for $n \geq 0$ via the adjunction morphism

$$p_{n+1\,*}p_{n+1}^* \simeq p_{n*}p_{n+1\,*}^j p_{n+1}^{j\,*}p_n^*\mathscr{F} \to p_{n*}p_{n+1\,*}^j \iota_{n*}^j \iota_n^{j*} p_{n+1}^{j\,*}p_n^*\mathscr{F} \simeq p_{n*}p_n^*\mathscr{F}.$$

Here we have used the identities $p_{n+1} = p_n \circ p_{n+1}^j$ and $p_{n+1}^j \circ \iota_{n+1}^j = \mathrm{id}$. This cosimplicial object "knows" about the sheaf Čech complex of $\mathscr{F}$ with respect to the covering $\mathfrak{U}$ which gave rise to our simplicial object $X_\bullet$, as we will see in Example 2.11

We wish to conclude our introductory discussion of simplicial objects in categories by focusing on the particularly interesting case in which $C$ is an abelian category. In this special case, there is a general construction (called the *Dold-Kan correspondence*) which effectively says that in an abelian category, a cosimplicial object is nothing more or less than a chain complex in non-negative degrees. One of the crucial applications of this correspondence for our purposes is that it will enable us to "see" certain aspects of the theory of injective resolutions (in the category of cosimplicial objects) which would otherwise be somewhat shrouded in mystery. More philosophically, the Dold-Kan correspondence shows that simplicial theory is a good non-abelian generalization of the chain complexes which are so useful in abelian categories.

Let $\mathscr{A}$ be an abelian category, and let $\mathrm{Simp}(\mathscr{A})$ denote the category of simplicial objects in $\mathscr{A}$. It is trivial to check that this is an abelian category, with formation of kernels and cokernels given by the termwise constructions within $\mathscr{A}$ with the obvious induced functoriality from $\Delta$ (i.e., induced face and degeneracy maps). In particular, we can check exactness by looking in each separate degree, and likewise for checking if a morphism is monic, epic, or an isomorphism. Also, observe that $\mathrm{Simp}(\mathscr{A})^{\mathrm{opp}} \simeq \mathrm{Cosimp}(\mathscr{A}^{\mathrm{opp}})$ in an evident manner.

There is a close connection between the category $\mathrm{Cosimp}(\mathscr{A})$ and the category $\mathrm{Ch}_{\geq 0}(\mathscr{A})$ of chain complexes in $\mathscr{A}$ concentrated in non-negative degrees:

**Definition 2.10.** We define the functor

$$\mathbf{s} : \mathrm{Cosimp}(\mathscr{A}) \to \mathrm{Ch}_{\geq 0}(\mathscr{A})$$

to assign to any cosimplicial object $C^\bullet$ the chain complex $\mathbf{s}(C^\bullet)$ whose degree $n$ term is $C^n$ and whose degree $n$ differential is

$$\partial_{C^\bullet}^n = \sum_{j=0}^{n+1}(-1)^j C^\bullet(\partial_j^{n+1})$$

for $n \geq 0$ (the relations on the $\partial_j$'s in $\Delta$ ensure that $\partial_{C^\bullet}^{n+1} \circ \partial_{C^\bullet}^n = 0$).

*Example* 2.11. Applying Definition 2.10 to Example 2.9 yields the classical sheaf Čech complex.

Of course, we have an obvious analogue of the functor $\mathbf{s}$ from simplicial objects to cochain complexes in nonnegative degrees, and we denote that functor $\mathbf{s}$ also without risk of confusion (though the formula defining the differential in degree $n$ now involves only $0 \leq j \leq n$ instead of $0 \leq j \leq n+1$).

It is trivial to check that $\mathbf{s}$ is an exact functor. Somewhat more remarkable is the (independent) discovery by Dold and Kan that inside of $\mathbf{s}(C^\bullet)$ is a functorial subcomplex $N(C^\bullet)$ (called the "normalized" chain complex associated to $C^\bullet$) which is not only quasi-isomorphic to $\mathbf{s}(C^\bullet)$ via the inclusion map but actual contains enough information to reconstruct the entire cosimplicial object $C^\bullet$! This may seem rather surprising, considering that we seem to have thrown out all information about the degeneracy maps when passing to $\mathbf{s}(C^\bullet)$, but the way around this is to realize that $\mathbf{s}(C^\bullet)$ contains a lot of redundant information in its terms $C^n$ in each degree (redundant in the sense that the many degeneracy maps into degree $n$ from higher degrees have images which usually overlap a lot, so we can't distinguish them just by being given $\mathbf{s}(C^\bullet)$). By passing to a well-chosen subcomplex with less redunancy, we will actually be able to reconstruct everythimg. This was the basic insight of Dold and Kan.

To illustrate the basic mechanism underlying the Dold-Kan correspondence, let's liven things up by working with a *simplicial* object $A_\bullet$ in $\mathscr{A}$ instead (once we state the general result, passing to the opposite category will give a result for cosimplicial objects too). For any simplicial object $A = A_\bullet$ in $\mathscr{A}$ and positive integer $n$, we define a subobject

$$N(A)_n = \bigcap_{j=1}^{n} \ker(\partial_n^j : A_n \to A_{n-1}).$$

Note that this intersection is taken over $1 \leq j \leq n$, not $0 \leq j \leq n$.

Define a differential $\mathrm{d}_n : N(A)_n \to N(A)_{n-1}$ via $A_\bullet(\partial_n^0)$ for $n \geq 1$. The relations on the $\partial$'s in $\Delta$ show that $\mathrm{d}_n$ really does take $N(A)_n$ into $N(A)_{n-1}$, and since $\mathrm{d}_n$ agrees with $\partial_{A,n}$ on $N(A)_n$ it follows that $\mathrm{d}_n \circ \mathrm{d}_{n+1}$ on $N(A)_{n+1}$ coincides with $\partial_{A,n} \circ \partial_{A,n+1} = 0$. By defining $N(A)_0 = A_0$ and $\mathrm{d}_0 = 0$, we have $\mathrm{d}_n \circ \mathrm{d}_{n+1} = 0$ for all $n \geq 0$, so $(N(A)_\bullet, \mathrm{d}_\bullet)$ is an ordinary cochain complex in $\mathscr{A}$ concentrated in non-negative degrees.

One easily checks that $N(A)_\bullet$ is naturally a subcomplex of $\mathbf{s}(A)$ which is moreover functorial in $A$. In fact, $N(A)_\bullet$ is even a direct summand subcomplex, with a complement in degree $n$ given by the sum of the images of the degeneracy maps from degrees *below* $n$ (recall $\mathbf{s}(A)_i = A_i$ for all $i$). The *proof* of Theorem 2.12 below makes it clear that this really gives a complementary subcomplex. One easily checks that $A \rightsquigarrow N(A)_\bullet$ defines an additive exact functor

$$N : \mathrm{Simp}(\mathscr{A}) \to \mathrm{CoCh}_{\geq 0}(\mathscr{A})$$

to the abelian category of cochain complexes in $\mathscr{A}$ concentrated in non-negative degrees. The miracle is that the abelian category structure will enable us to reconstuct the simplicial object $A$ just from the data of the cochain complex $N(A)$, and in fact the functor $N$ will turn out to be an equivalence of categories (with a precise quasi-inverse functor).

Let's see what is going on in low degrees. In degree 1, since $\partial_1^j \sigma_0^j$ is the identity on $A_1$ we see that $\sigma_0^0 : A_0 \to A_1$ is naturally *split*, and in fact one can show from the simplicial map relations that the natural map

$$N(A)_1 \oplus N(A)_0 \to A_1, \tag{2.5}$$

with $N(A)_0$ mapped in via $\sigma_0^0$, is an isomorphism (Theorem 2.12 below tells us much more). Under this identitification, the map $\sigma_0^0 : A_0 \to A_1$ is explicated by the inclusion of $N(A)_0$ into the left side of (2.5), and the map $\partial_j^0 : A_1 \to A_0$ is given by projection onto $N(A)_0$ for $j = 1$ and is given by the sum of this projection map and $\mathrm{d}_1 : N(A)_1 \to N(A)_0 = A_0$ for $j = 0$. In this way, the truncation of $A_\bullet$ in degrees $\leq 1$ is reconstructed from $N(A)_\bullet$ in degrees $\leq 1$. A clever inductive iteration of this to higher degrees yields the Dold-Kan correspondence:

**Theorem 2.12.** (Dold-Kan) *The exact functor $N : \mathrm{Simp}(\mathscr{A}) \to \mathrm{CoCh}_{\geq 0}(\mathscr{A})$ is an equivalence of categories. Moreover, the complex $N(A)$ is naturally direct summand of the cochain complex $\mathbf{s}(A)$ with the inclusion inducing an isomorphism on homology.*

The evident analogue holds for the categories of cosimplicial objects and chain complexes (in non-negative degrees), and there is an analogue of the "normalized (co)chain complex" construction which goes over essentially the same way. We have formulated the theorem for simplicial objects since this is the version handled in the reference we give for the proof below; of course, passage to the opposite category renders the two situations equivalent.

*Proof.* See [Mac, pp.11–13] (in which simplicial complexes are strangely called FD-complexes) for a "splitting" construction for objects in $\mathrm{Simp}(\mathscr{A})$, as well as the application of this "splitting" to motivate and study the construction of a quasi-inverse functor $T$ to $N$, with $T(C_\bullet)_n$ defined as a somewhat involved finite direct sum of various terms $C_m$ with $m \leq n$, with components of the direct sum indexed by various degeneracy maps among $[m]$'s for $m \leq n$. Lemma 2.2 is useful for the task of actually constructing the well-defined simplicial structure maps between the $T(C_\bullet)_n$'s. This construction makes explicitly clear that $N(A)$ is a direct summand of $\mathbf{s}(A)$ with complementary *complex* $D(A)$ given in degree $n$ by the sum of images of degeneracy maps from lower degrees (one can use relations as in Lemma 2.2 to see a priori that the $D(A)_n$'s constitute a subcomplex of $\mathbf{s}(A)$). Strictly speaking, the proof in [Mac] is given with $\mathscr{A}$ the category of abelian groups, but by chasing "points" one sees that the same argument is valid in an arbitrary abelian category.

See [W, pp. 266-7] for the quasi-isomorphism property of the inclusion $N(A) \to \mathbf{s}(A)$. One just has to show that $D(A)$ is acyclic, and this is done by filtering $D(A)$ and proving that the successive quotients are acyclic. Beware that the calculations in the proof of acyclicity in [W, Thm 8.3.8] are riddled with sign errors. ■

Keeping in mind that the sheaf example from Example 2.8 was nearly an example of a cosimplicial object, the following corollary of the *proof* of the Dold-Kan correspondence will wind up playing an important technical role in the study of sheaves on simplicial objects later on.

**Corollary 2.13.** *Let $\mathscr{A}$ be an abelian category with enough injectives. The category* $\mathrm{Cosimp}(\mathscr{A})$ *has enough injectives. In fact, a cosimplicial object $C^\bullet$ in $\mathscr{A}$ is injective in* $\mathrm{Cosimp}(\mathscr{A})$ *if and only if each $C^n$ is an injective object in $\mathscr{A}$ and the associated chain complex* $\mathbf{s}(C^\bullet)$ *in non-negative degrees is acyclic in positive degrees.*

Of course, in the sheaf context we will be interested in "cosimplicial" objects and injective resolutions, and not projective resolutions, so we expect to be more interested in projective resolutions only in the context of simplicial objects.

By passage to opposite categories, we get a similar result concerning projectives in the category $\mathrm{Simp}(\mathscr{A})$. This corollary is a prototype for a later result (Lemma 6.4) which will tell us about the structure of injective objects in the category of "sheaves" on a simplicial topological space (or scheme, or object of any site), and that in turn will be used in the construction of the cohomological descent spectral sequence in Theorem 6.11.

*Proof.* By the Dold-Kan correspondence, the problem of having enough injectives in $\mathrm{Cosimp}(\mathscr{A})$ is equivalent to the problem of having enough injectives in $\mathrm{Ch}_{\geq 0}(\mathscr{A})$. But it is a basic fact in homological algebra that when $\mathscr{A}$ has enough injectives then so does $\mathrm{Ch}_{\geq 0}(\mathscr{A})$, and moreover the injectives in this latter category are precisely the chain complexes $C^\bullet$ with $C^n$ an injective of $\mathscr{A}$ for each $n$ and $\mathrm{H}^i(C^\bullet) = 0$ for all $i > 0$.

It remains to describe the injective objects in $\mathrm{Cosimp}(\mathscr{A})$. Well, if $C^\bullet$ is injective in here, then $N(C^\bullet)$ is an injective in $\mathrm{Ch}_{\geq 0}(\mathscr{A})$. That is, $N(C^\bullet)^n$ is an injective in $\mathscr{A}$ for each $n$ and $\mathrm{H}^i(N(C^\bullet)) = 0$ for all $i > 0$. But the quasi-inverse functor in the Dold-Kan correspondence constructs each $C^n$ as a finite direct sum of $N(C^\bullet)^m$'s for various $m \leq n$ (with some terms repeated several times). Since a finite direct sum of injectives is injective, we conclude that if $C^\bullet$ is injective in $\mathrm{Cosimp}(\mathscr{A})$ then $C^n$ is injective in $\mathscr{A}$ for each $n$ and moreover

$$\mathrm{H}^i(\mathbf{s}(C^\bullet)) \simeq \mathrm{H}^i(N(C^\bullet)) = 0$$

for all $i > 0$, the isomorphism coming from Theorem 2.12. Conversely, if $C^\bullet$ is a cosimplicial object with each $C^n$ injective in $\mathscr{A}$ and $\mathbf{s}(C^\bullet)$ acyclic in positive degrees, we want to show that $C^\bullet$ is an injective object in $\mathrm{Cosimp}(\mathscr{A})$. By the Dold-Kan correspondence, we want $N(C^\bullet)$ to be injective in $\mathrm{Ch}_{\geq 0}(\mathscr{A})$, which is to say that each $N(C^\bullet)^n$ should be an injective in $\mathscr{A}$ and $N(C^\bullet)$ should be acyclic in positive degrees. Since $N(C^\bullet) \to \mathbf{s}(C^\bullet)$ is a quasi-isomorphism, the acyclicity is clear. Since each $N(C^\bullet)^n$ is a direct summand of the injective $C^n$, it follows that $N(C^\bullet)^n$ is injective in $\mathscr{A}$ for each $n \geq 0$. ■

## 3. Coskeleta

Having survived the arid generality of (co)simplicial objects, we are ready to see how that theory provides the context for carrying out a vast generalization of Čech theory via the simplicial concept of hypercoverings. We have seen already how Čech theory can be recast as a special instance of (augmented) simplicial topology. Hypercovers will be certain *special* kinds of augmented simplicial spaces which satisfy a more subtle property enjoyed by the fiber power Čech construction, going beyond the mere simplicial structure.

In order to even *define* what a hypercovering is, and to make sense of "refining" a simplicial object in higher degrees without affecting lower degrees (something we cannot conceive in the framework of ordinary Čech theory, but which underlies what is so "hyper" about hypercovers – see Theorem 4.13), we need the coskeleton functors. For example, just as an ordinary covering has a certain kind of "surjectivity" condition built into the definition, a hypercovering will be an augmented simplicial object satisfying an analogous "surjectivity" requirement in each degree, expressed in terms of certain coskeleta.

As another important application of the concepts we are about to introduce, we mention that coskeleta will give us a technique for proving theorems about general simplicial concepts (such as hypercovers) by means of induction on degree. A dramatic application of such inductive arguments is the use of the (topological or étale) proper base change theorem for "sheaves in degree 0" to create the theory of cohomological descent on the level of derived categories of "sheaves" on proper hypercoverings. In a nutshell, the theory of hypercoverings will be a "derived" version of the classical theory of coverings, with the cohomological descent property of proper hypercoverings serving as a "derived" version of the proper base change theorem for ordinary sheaves.

Without further delay, let us prepare for the definition and basic properties of coskeleta. Just as we want to have the intuition of thinking of a simplicial object $X_\bullet$ as a generalized CW-complex, we can think of the truncated version (as if we stopped gluing cells at dimension $n$). More specifically, if we let $\Delta_{\leq n}$ and $\Delta^+_{\leq n}$ denote the full subcategories of $\Delta$ and $\Delta^+$ consisting of objects $[m]$ with $m \leq n$, we can make the:

**Definition 3.1.** For $n \geq 0$, an *n-truncated simplicial object* in a category $C$ is a contravariant functor $X_\bullet : \Delta_{\leq n} \to C$. Such objects form a category $\mathrm{Simp}_n(C)$, with $\mathrm{Simp}_n(C)$ abelian if $C$ is (using the evident term-by-term constructions for kernels and cokernels).

We define the augmented and cosimplicial variants in the obvious analogous manner (using $\Delta^+_{\leq n}$, etc.).

One has an evident analogue of Lemma 2.2 for describing the category $\Delta_{\leq n}$ in terms of generators and relations, and the same proof carries over verbatim. Consequently, as with (co)simplicial objects, we can describe $n$-truncated (co)simplicial objects (as well as augmented ones) in terms of a more minimalist amount of data by just specifying face and degeneracy (and augmentation) maps subject to the expected relations.

Given a simplicial object $X_\bullet : \Delta \to C$ we can "restrict" it to the full subcategory $\Delta_{\leq n}$ (and similarly with the augmented version), and in this way we get a functor from (co)simplicial objects in $C$ to $n$-truncated (co)simplicial objects in $C$ (and similarly if we impose augmentations). We write

$$\mathrm{sk}_n : \mathrm{Simp}(C) \to \mathrm{Simp}_n(C)$$

for this $n$-truncation functor (and use a similar notation on categories of augmented objects). We will call $\mathrm{sk}_n(X_\bullet)$ the *n-skeleton* of $X_\bullet$, but beware that this is not always what is called the $n$-skeleton of $X_\bullet$ (this terminology is sometimes used to describe a genuine simplicial object − not a truncated one − which is "generated" by what we have called $\mathrm{sk}_n(X_\bullet)$ by formally adjoining simplices in degrees $> n$ by means of "applying" degeneracy maps to the $X_m$'s for $m \leq n$ in a suitably universal manner). Using $\Delta^+_{\leq n}$ we likewise get truncation functors

$$\mathrm{sk}_n : \mathrm{Simp}^+(C) \to \mathrm{Simp}^+_n(C)$$

for $n \geq -1$ (with $\mathrm{sk}_{-1}$ just the assignment of the augmented object in degree $-1$ in $\mathrm{Simp}^+_{-1}(C) = C$). We should probably write $\mathrm{sk}^+_n$, but we won't.

The problem which is "solved" by the $n$-coskeleton construction is to determine whether $\mathrm{sk}_n$ has a right adjoint:

**Definition 3.2.** For $n \geq 0$, an *n-coskeleton* functor $\mathrm{cosk}_n : \mathrm{Simp}_n(C) \to \mathrm{Simp}(C)$ is a functor for which there is a bifunctorial bijection

$$\mathrm{Hom}_{\mathrm{Simp}_n(C)}(\mathrm{sk}_n(X_\bullet), Y_\bullet) \simeq \mathrm{Hom}_{\mathrm{Simp}(C)}(X_\bullet, \mathrm{cosk}_n(Y_\bullet))$$

for any $n$-truncated cosimplicial object $Y_\bullet$ in $C$ (and similarly in the augmented case with $n \geq -1$ and augmented simplicial object categories.).

We should probably use notation $\mathrm{cosk}^+_n$ in the augmented case, but we won't; this seems to present little risk of confusion, though one must be careful to note that $\mathrm{cosk}_n$ does *not* commute with the "forget" functors $\mathrm{Simp}^+_n(C) \to \mathrm{Simp}_n(C)$ and $\mathrm{Simp}^+(C) \to \mathrm{Simp}(C)$, for much the same reason that forming the underlying scheme of a fiber product over a base is not the same as forming the product of the underlying "bare" schemes (see Example 3.3 below). If $Y'_\bullet = Y_\bullet/S$ is an $n$-truncated augmented simplicial object, we sometimes denote the augmented simplicial space $\operatorname{cosk}_n(Y'_\bullet)$ in other ways: $\operatorname{cosk}_n(Y_\bullet/S)$ or $\operatorname{cosk}_n(Y_\bullet/S) \to S$. In particular, when we choose the latter option we are intending $\operatorname{cosk}_n(Y_\bullet/S)$ to denote just the part of the coskeleton in non-negative degrees. The context should always make clear what is intended.

Of course, we have to prove (under suitable conditions on $C$) that an $n$-coskeleton functor exists. In more concrete terms, given an $n$-truncated $Y_\bullet$, to compute $\operatorname{cosk}_n(Y_\bullet)$ we must determine the "stuff" to introduce in higher degrees so that any map $f : \operatorname{sk}_n(X_\bullet) \to Y_\bullet$ *uniquely* "extends" to a map $X_\bullet \to \operatorname{cosk}_n(Y_\bullet)$ We will see in Corollary 3.10 that in fact the adjoint $\operatorname{cosk}_n$ has the additional property that the adjunction map $\operatorname{sk}_n(\operatorname{cosk}_n(Y_\bullet)) \to Y_\bullet$ is an isomorphism in favorable situations for which we can show $\operatorname{cosk}_n$ exists in the first place, so thinking of $\operatorname{cosk}_n(Y_\bullet)$ as a "higher degree" extension of $Y_\bullet$ is reasonable. That said, what we really must do is construct for each $Y_\bullet$ in $\operatorname{Simp}_n(C)$ a simplicial object $\operatorname{cosk}_n(Y_\bullet)$ and an adjunction map $\operatorname{sk}_n(\operatorname{cosk}_n(Y_\bullet)) \to Y_\bullet$ making the composite map

$$\operatorname{Hom}_{\operatorname{Simp}(C)}(X_\bullet, \operatorname{cosk}_n(Y_\bullet)) \xrightarrow{\operatorname{sk}_n} \operatorname{Hom}_{\operatorname{Simp}_n(C)}(\operatorname{sk}_n(X_\bullet), \operatorname{sk}_n(\operatorname{cosk}_n(Y_\bullet))) \to \operatorname{Hom}_{\operatorname{Simp}_n(C)}(\operatorname{sk}_n(X_\bullet), Y_\bullet)$$

bijective for any $X_\bullet$. Once again, we must also keep in mind the augmented case (after all, Čech theory gives rise to augmented simplicial objects).

Since we will show later that (in favorable cases) $\operatorname{sk}_n(\operatorname{cosk}_n(Y_\bullet)) \to Y_\bullet$ is an isomorphism, we can safely imagine the universal property of $\operatorname{cosk}_n(Y_\bullet)$ as being a simplicial object extending $Y_\bullet$ such that for any simplicial object $X_\bullet$ and map $f : \operatorname{sk}_n(X_\bullet) \to Y_\bullet$ in degrees $\leq n$, the map $f$ admits a *unique* extension $F : X_\bullet \to \operatorname{cosk}_n(Y_\bullet)$ on the level of simplicial objects (and likewise in the augmented case).

We will study the special case $n = 0$ in a moment (before we treat the general case under some mild auxiliary hypothesis on $C$), but we first want to stress that the idea on the geometric side is that $\operatorname{cosk}_n(Y_\bullet)$ has the "least" amount of simplicial data in higher degrees which is necessary given that we are starting with $Y_\bullet$ in degrees $\leq n$ and that for any map $\operatorname{sk}_n X_\bullet \to Y_\bullet$ there must be somewhere for each $X_m$ (with $m > n$) to go inside of $\operatorname{cosk}_n Y_\bullet$.

For example, given a topological 1-simplex $Y_\bullet$ which is the edge $E$ of a triangle $T$ we could imagine in $(\operatorname{cosk}_1 Y_\bullet)_2$ that we are forced to have that triangle $T$ put in as a 2-cell since an actual triangle whose 1-truncation maps via the identity to $E$ can only map to $\operatorname{cosk}_1(Y_\bullet)$ in degree 2 if the triangle 2-cell in the source has somewhere to go! However, we cannot forget that not only are we required to be able to lift maps to $Y_\bullet$ in degrees $\leq n$ to maps all the way up to every level on the $n$-coskeleton $\operatorname{cosk}_n(Y_\bullet)$, but such a lifting has to be *unique*. This makes it more difficult to visualize what $\operatorname{cosk}_n(Y_\bullet)$ looks like if we're trying to think intuitively in terms of ordinary topological categories (as opposed to homotopy categories!). For this reason, I don't actually try to visualize the $n$-coskeleton at all beyond the vague intuition which has just been described; it is the *functorial property* of $\operatorname{cosk}_n$ (as being right adjoint to $\operatorname{sk}_n$) which is what matters for our purposes.

*Example* 3.3. Let's now consider the case $n = 0$ and construct $\operatorname{cosk}_0$ "by hand" before we attack the general case. In fact, the case $n = 0$ is theoretically important both for the role it plays in starting inductive arguments (for which having a concrete description of $\operatorname{cosk}_0$ is quite useful!) and also because it is the precise point of contact between Čech theory and the theory of hypercoverings. In essence, as we shall see, Čech theory is part of the theory of $\operatorname{cosk}_0$ in the augmented case.

First consider the non-augmented case. Given an object $Y_0$ in $C$, is there some "final" simplicial object $X_\bullet$ in $C$ with respect to the property of being equipped with a map $\phi_0 : X_0 \to Y_0$ in $\operatorname{Simp}_0(C) = C$? That is, we seek such a pair $(X_\bullet, \phi_0)$ such that for *any* simplicial object $X'_\bullet$ equipped with a map $\phi'_0 : X'_0 \to Y_0$ there is a unique map $\xi : X'_\bullet \to X_\bullet$ with $\phi_0 \circ \xi_0 = \phi'_0$. We will carry out the construction of such a pair $(X_\bullet, \phi_0)$ under the hypothesis that $C$ admits finite products, and we will see that Example 2.4 with $\phi_0 = \operatorname{id}_{Y_0}$ provides the universal solution.

Given $\phi'_0 : X'_0 \to Y_0$, we get two maps

$$X'_1 \underset{\mathrm{d}_1^1}{\overset{\mathrm{d}_1^0}{\rightrightarrows}} X'_0 \xrightarrow{\phi'_0} Y_0$$

and hence we get a unique map

$$\xi_1' : X_1' \to Y_0 \times Y_0$$

such that the diagram

$$\begin{array}{ccc} X_1' & \xrightarrow{\xi_1'} & Y_0 \times Y_0 \\ {\scriptstyle d_1^0}\downarrow\downarrow{\scriptstyle d_1^1} & & {\scriptstyle p_0}\downarrow\downarrow{\scriptstyle p_1} \\ X_0' & \xrightarrow[\phi_0']{} & Y_0 \end{array}$$

commutes. Also, compatibility with the section from degree 0 is straightfoward (using the diagonal section on the right side). Similarly, composing $\phi_0'$ with the *three* composite "face" maps $X_2' \to X_0'$ (corresponding to the three maps $[0] \to [2]$ in $\Delta$) yields a map

$$\xi_2' : X_2' \to Y_0 \times Y_0 \times Y_0$$

such that the diagram

$$\begin{array}{ccc} X_2' & \xrightarrow{\xi_2'} & Y_0 \times Y_0 \times Y_0 \\ {\scriptstyle d_2^0}\downarrow\downarrow\downarrow{\scriptstyle d_2^2} & & {\scriptstyle p_{12}}\downarrow\downarrow\downarrow{\scriptstyle p_{01}} \\ X_1' & \xrightarrow[\xi_1']{} & Y_0 \times Y_0 \end{array}$$

commutes, with middle vertical arrows $d_2^1$ and $p_{02}$ (the commutativity is checked by composing both ways around with the two projections from $Y_0 \times Y_0$ to $Y_0$). This commutativity uniquely determines $\xi_2'$, and it is easy to check that we also have *compatibility with the degeneracies* from degree 1 to degree 2. Continuing, one finds quite easily that the "Čech cover" $X_n = Y_0^{\times(n+1)}$ with evident simplicial structure as in Example 2.4 and the evident

$$\phi_0 : X_0 = Y_0 \stackrel{\text{id}}{\to} Y_0$$

serves as $\text{cosk}_0(Y_0)$ in $C$.

The augmented version goes essentially the same way, but we must assume $C$ admits finite fiber products. If we're given some object $Y_0 \to Y_{-1}$ in $\text{Simp}_0^+(C)$, then we can either see that it suffices to focus on the slice category $C_{/Y_{-1}}$ of objects over $Y_{-1}$ (in which products are just fibers products in $C$ over $Y_{-1}$), thereby reducing to the previous case, or we can just directly go through the preceding argument in the augmented situation. Such a direct argument shows that essentially the same construction works, the only difference being that we must use fiber products over $Y_{-1}$ (as opposed to absolute products in $C$) so as to keep track of compatibility with augmentations. This makes it clear that Čech theory is part of the theory of $\text{cosk}_0$ for augmented simplicial objects.

*Example* 3.4. Let's try to do the non-augmented $n = 1$ case by hand, assuming $C$ to contain finite products and finite fiber products. We'll pretty quickly see that this begins to look like a combinatorial mess. We give ourselves a 1-truncated simplicial object

$$d_0, d_1 : Y_1 \to Y_0, \;\; s_0 : Y_0 \to Y_1$$

with $s_0$ a section to both $d_j$'s. We also give ourselves a simplicial object $X_\bullet$ and a map $\phi : \text{sk}_1 X_\bullet \to Y_\bullet$ in degrees $\leq 1$. That is, we give ourselves maps $\phi_1 : X_1 \to Y_1$ and $\phi_0 : X_0 \to Y_0$ compatible with the face and degeneracy maps between degrees 0 and 1.

Let's try to figure out what the degree 2 term $Y_2$ of $\text{cosk}_1(Y_\bullet)$ should be (not even worrying about how to explicate the face and degeneracy maps between $Y_2$ and $Y_1$). Using the three face maps $X_2 \to X_1$ and composing with $\phi_1$ we get three maps $X_2 \to Y_1$. However, the resulting map

$$X_2 \to Y_1 \times Y_1 \times Y_1$$

is not arbitrary. For example, since $\phi_1$ and $\phi_0$ are compatible with the face maps to degree 0, the three maps $X_2 \to Y_1$ enjoy certain extra relations for their composites with the two face maps $Y_1 \to Y_0$ to degree

0. This translates into the map to the triple product of $Y_1$ landing inside a fiber product subobject of the form

$$Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1$$

where each of the four implicit projections $Y_1 \to Y_0$ is one of the two face maps (determined by thinking about the case in which $Y_1 = Y_0 \times Y_0$ and trying to ensure that this triple fiber product is exactly just the triple product of $Y_0$ without "re-ordering" the factors). We omit the explication, because already we see that more work is needed: the resulting map

$$X_2 \to Y_1 \times_{Y_0} Y_1 \times_{Y_0} Y_1$$

enjoys yet more properties because of the requirement that $\phi$ in degrees $\leq 1$ respects the degeneracy map from degree 0 to degree 1.

It gets very complicated to write down explicitly! In fact, this sort of hands-on approach to figuring out what $\mathrm{cosk}_1(Y_\bullet)$ should be in degree 2 involves having to think about commutative diagrams among maps $[m] \to [2]$ with $m = 0, 1$, and there are a *lot* of such diagrams! In fact, it turns out that there is much redundancy and one only really needs to think about the six such maps $[m] \to [2]$ with $m \leq 1$ which are injective, but already it becomes quite painful to think about these issues in such an explicit manner. One thing we do see, however, is that the three maps $X_2 \to Y_1$ have to satisfy a variety of compatibility conditions with respect to maps between $Y_1$ and $Y_0$. This has the feeling of requiring that we specify a map from $X_2$ to some "inverse limit" of a suitable diagram (with many maps) involving the objects $Y_1$ and $Y_0$, and it is exactly the point of view of inverse limits over finite diagrams which will resolve the general case in a way that avoids much pain.

In the general case, to construct coskeletons we will assume that the category $C$ admits finite fiber products, or equivalently inverse limits over finite diagrams. Let us recall what limits over a diagram mean, as this will be used in the *construction* of $\mathrm{cosk}_n$.

Let $D$ be a small category and $F : D \to C$ be a *contravariant* functor. For example, if $D$ were a directed set (viewed as a category in an evident manner) then $F$ would just be the specification of an inverse system in $C$. In general, we will need to allow non-identity morphisms in $D$. What we want to do is think about $F(D)$ as a "diagram" in $C$ and we wish to construct an object in $C$ which is endowed with maps to all of the $F(d)$'s for objects $d$ of $D$, compatibly with $F(f)$'s for all morphisms $f$ in $D$, and which is universal with respect to this property. This is made precise by:

**Definition 3.5.** With notation as above, we define an *inverse limit*

$$\varprojlim_D F$$

of $F$ over $D$ to be an object $L$ of $C$ equipped with maps $\phi_d : L \to F(d)$ for each object $d$ of $D$ such that

- $\phi_d = F(f) \circ \phi_{d'}$ for all $f \in \mathrm{Hom}_D(d, d')$,
- for any object $X$ of $C$ equipped with maps $\psi_d : X \to d$ for all objects $d$ of $D$ such that $\psi_d = f \circ \psi_{d'}$ for all $f \in \mathrm{Hom}_D(d, d')$, there is a unique map $\xi : X \to L$ in $C$ such that $\psi_d = \psi_d \circ \xi$ for all objects $d$ of $D$.

If $F$ is covariant, we have a similar definition of inverse limit by working instead with maps $f \in \mathrm{Hom}_D(d', d)$ in the above map relations. The corresponding notion of *direct limit* for $F : D \to C$ is defined similarly.

Our interest will be in taking inverse limits when $D$ is a *finite* category, by which we mean that $D$ has finitely many objects and finitely many morphisms among them.

*Example* 3.6. Given the diagram

$$X' \to X \leftarrow X'',$$

an inverse limit is just a fiber product $X' \times_X X''$, while an inverse limit over a diagram consisting of only two objects with just identity morphisms is an ordinary product.

*Example* 3.7. If $D$ is a finite category and $C$ admits finite fiber products and products, then the inverse limit of $F$ over $D$ is naturally a subobject of the product of the finitely many $F(d)$'s (for reasons very much like in the category of sets).

In fact, by carefully inducting on the "size" of a finite $D$, we see that an inverse limit in the sense of the preceding definition will exist whenever $D$ is finite and $C$ admits the existence of finite products and finite fiber products (i.e., the two examples considered in Example 3.6). We describe this condition by saying that *$C$ admits finite inverse limits.*

**Lemma 3.8.** *If $C$ admits finite inverse limits, so do the categories* $\mathrm{Simp}(C)$, $\mathrm{Simp}^+(C)$, $\mathrm{Simp}_n(C)$, *and* $\mathrm{Simp}_n^+(C)$. *The same goes for categories of augmented objects.*

*Proof.* One does constructions in $C$ in each separate degree and checks that the output has the desired universal property in the various categories of interest. ■

We can now state the basic existence result for coskeleta.

**Theorem 3.9.** *Assume $C$ admits finite inverse limits (or equivalently, admits finite products and finite fiber products). Then the functor $\mathrm{sk}_m : \mathrm{Simp}(C) \to \mathrm{Simp}_m(C)$ admits a right adjoint $\mathrm{cosk}_m$ for all $m \geq 0$. The same statement holds for the categories of augmented objects, then also allowing $m = -1$.*

Although we have already settled the case $m = 0$ in the non-augmented situation in Example 3.3, we will see that our general argument recovers our explication of $\mathrm{cosk}_0$. Knowledge of how $\mathrm{cosk}_m$ is constructed will be used in some later proofs!

*Proof.* Since $C$ admits fibers products, for the augmented case it suffices by a pullback argument to restrict to the case of a slice category $C_{/S}$ (which also admits finite fiber products), and to consider only objects augmented by $S$. But then aside from the case $m = -1$ this brings the augmented case down to the non-augmented case (using the category $C_{/S}$ instead of $C$). We first quickly dispose of the augmented case with $m = -1$ and then we will just have to treat the case of categories of simplicial (and truncated simplicial) objects without augmentation.

For $m = -1$, the right adjoint to the functor $\mathrm{sk}_{-1} : \mathrm{Simp}^+(C) \to C$ is the "constant augmented complex" functor which assigns to each object $Y_{-1}$ in $C$ the augmented simplicial complex given by $Y_{-1}$ in each degree with all simplicial maps (including augmentation) given by the identity map.

Now we turn to the case of (non-augmented) simplicial objects and fix $m \geq 0$ and fix an $m$-truncated simplicial object $Y = Y_\bullet$ in $C$. We will construct a simplicial object which should be $\mathrm{cosk}_m Y$, and then will see that it enjoys the expected properties.

The first order of business is to construct what should be the degree $n$ piece of $\mathrm{cosk}_m Y$ before we worry about making it functorial in $[n]$. To this end, fix $n \geq 0$. Consider the "representable" *simplicial* complex of sets

$$\Delta[n] : [k] \rightsquigarrow \mathrm{Hom}_\Delta([k], [n]),$$

to be viewed as a combinatorial abstraction of the classical $n$-simplex $\Delta[n]_{\mathbf{R}}$ in $\mathbf{R}^{n+1}$ (with contravariant functorial structure in $[k]$ given in the obvious manner). We view $\Delta[n]$ as a category with objects given by the elements in each $\Delta[n]_k$ (i.e., objects are maps $\phi : [k] \to [n]$) and a morphism from $\phi : [k] \to [n]$ to $\phi' : [k'] \to [n]$ is a map $\alpha : [k] \to [k']$ such that $\phi' \circ \alpha = \phi$. In other words, $\Delta[n]$ as a category is just the slice category $\Delta_{/[n]}$ of objects in $\Delta$ over $[n]$.

Note that applying Yondea's lemma in the category $\Delta$ yields a bijection

$$\mathrm{Hom}_\Delta([n'], [n]) \simeq \mathrm{Hom}_{\mathrm{Simp}(\mathrm{Set})}(\Delta[n'], \Delta[n])$$

for any $n, n' \geq 0$. We likewise define the $m$-truncated simplicial complex of sets $\mathrm{sk}_m(\Delta[n])$, with the underlying set $\Delta[n]_k$ in degree $k \leq m$ consisting of the set of maps $[k] \to [n]$ in $\Delta$. Note that even when $m > n$ this can can be quite big. We view $\mathrm{sk}_m \Delta[n]$ as a full subcategory of $\Delta[n]$. Note that this is a finite category.

In the augmented case, we can define $\Delta^+[n]$ similarly (and interpret it as an abstract of the idea of a pointed $n$-simplex), and we likewise get the finite category $\mathrm{sk}_m \Delta^+[n]$. Note that we can even make sense of this latter construction for any $m, n \geq -1$, and when $n = -1$ these categories are all just a single object with only the identity morphism (since there is a unique arrow in $\Delta$ with target $[-1] = \emptyset$, namely the identity map of $[-1]$.) This is related to our earlier observation that $\operatorname{cosk}_{-1}$ is just the "constant (augmented) simplicial object" construction.

Returning back to the non-augmented situation, consider the finite category $\mathrm{sk}_m(\Delta[n])$. For each object $\phi : [k] \to [n]$ in $\mathrm{Hom}([k],[n]) = \Delta[n]_k$ with $k \le m$, define $Y_\phi = Y_k$. Given another $\phi' : [k'] \to [n]$ with $k' \le m$ and a morphism $\alpha : \phi \to \phi'$ in $\mathrm{sk}_m(\Delta[n])$ (i.e., a morphism $\alpha : [k] \to [k']$ in $\Delta$ such that $\phi' \circ \alpha = \phi$) we get a morphism $Y(\alpha) : Y_{k'} \to Y_k$, or equivalently a morphism

$$Y(\alpha) : Y_{\phi'} \to Y_\phi.$$

As we vary over the $\phi$'s and $\alpha$'s, we have made a new contravariant functor $\mathrm{sk}_m \Delta[n] \to C$ which we denote $Y$. It makes sense to form the inverse limit object for this situation:

$$\widetilde{Y}_n^{(m)} = \varprojlim_{\mathrm{sk}_m(\Delta[n])} Y_\phi.$$

This object captures all relations "among" $[k]$'s for $k \le m$, at least as far as their maps to $[n]$ are concerned, and essentially for this reason $\widetilde{Y}_n^{(m)}$ will turn out to be the degree $n$ part of the $m$-coskeleton of the given $m$-truncated object $Y_\bullet$. Of course, making sense of this requires that we first must make $[n] \rightsquigarrow \widetilde{Y}_n^{(m)}$ into a contravariant functor $\Delta \to C$. If $m = 0$, the category $\mathrm{sk}_m(\Delta[n])$ is a discrete category with $n+1$ points, so $\widetilde{Y}_n^{(m)}$ is just the product $Y_0^{\times(n+1)}$.

Now we explicate the functoriality of $[n] \rightsquigarrow \widetilde{Y}_n^{(m)}$ (e.g., for $m = 0$ we expect the usual Čech-like cartesian power simplicial object). For any morphism $\alpha : [n'] \to [n]$, we want to define a map $\widetilde{Y}_n^{(m)} \to \widetilde{Y}_{n'}^{(m)}$ arising from $\alpha : [n'] \to [n]$ by "contravariance." To define this, we will define maps from $\widetilde{Y}_n^{(m)}$ to each object appearing in the diagram over $\mathrm{sk}_m(\Delta[n'])$ whose inverse limit is $\widetilde{Y}_{n'}^{(m)}$, and we will then check the necessary compatibilities to ensure that these maps glue to give a map from $\widetilde{Y}_n^{(m)}$ to the inverse limit $\widetilde{Y}_{n'}^{(m)}$.

For any object $\phi' : [k'] \to [n']$ in $\mathrm{sk}_m(\Delta[n'])$ (with $k' \le m$) we get a natural map

$$\xi_\phi : \widetilde{Y}_n^{(m)} \to Y_{\phi'} = Y_{k'} \tag{3.1}$$

by canonically projecting the inverse limit $\widetilde{Y}_n^{(m)}$ over $\mathrm{sk}_m(\Delta[n])$ onto $Y_{\alpha\circ\phi'} = Y_{k'}$ (note that $\alpha \circ \phi'$ lies in the category $\mathrm{sk}_m \Delta[n]$). The maps $\xi_{\phi'}$ in (3.1) are natural in $\phi'$ in the sense that for any commutative diagram

$$\begin{array}{ccc} [k'_1] & \xrightarrow{\phi'_1} & [n'] \\ {\scriptstyle\beta}\downarrow & \nearrow{\scriptstyle\phi'_2} & \\ [k'_2] & & \end{array}$$

we get a diagram

$$\begin{array}{ccccc} \widetilde{Y}_n^{(m)} & \xrightarrow{\xi_{\alpha\circ\phi'_1}} & Y_{\alpha\circ\phi'_1} & = & Y_{k'_1} \\ & {\scriptstyle\xi_{\alpha\circ\phi'_2}}\searrow & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle Y(\beta)} \\ & & Y_{\alpha\circ\phi'_2} & = & Y_{k'_2} \end{array} \tag{3.2}$$

in which the left side commutes by the universal property of $\widetilde{Y}_n^{(m)}$ as an inverse limit, the right square makes sense since $k'_1, k'_2 \le m$, and the right square commutes by the very definition of $Y_\phi$ for $\phi$ in $\mathrm{sk}_m \Delta[n]$ and how $\phi \rightsquigarrow Y_\phi$ is functorial in $\phi$.

By the commutativity of the outer edge of (3.2), we conclude that (3.1) is natural in $\phi'$, so we may pass to the inverse limit over $\phi'$'s and conclude that the data in (3.1) uniquely determine a morphism

$$\widetilde{Y}^{(m)}(\alpha) : \widetilde{Y}_n^{(m)} \to \widetilde{Y}_{n'}^{(m)}$$

in $C$. In other words, for each morphism $\alpha : [n'] \to [n]$ in $\Delta$ we have constructed a morphism $\widetilde{Y}^{(m)}(\alpha) : \widetilde{Y}_n^{(m)} \to \widetilde{Y}_{n'}^{(m)}$. It is trivial to check (by projection to "pieces" of an inverse limit) that this construction is contravariant with respect to composition in $\alpha$, so we have defined a simplicial object $\widetilde{Y}^{(m)}$ in $C$ which might might suggestively write as

$$\widetilde{Y}^{(m)} = \varprojlim_{\mathrm{sk}_m \Delta} Y_\phi$$

(with the understanding that the object $\widetilde{Y}^{(m)}_n$ in degree $n$ is an inverse limit over $\mathrm{sk}_m \Delta[n]$). One easily checks that when $m = 0$ we get exactly the simplicial object on cartesian powers of $Y_0$ with *exactly* the expected functorial structure on $\Delta$ (i.e., we get our explication of $\mathrm{cosk}_0$ as in Example 3.3, or really as in Example 2.4).

Before we can show that $\widetilde{Y}^{(m)}$ can serve as $\mathrm{cosk}_m Y$, we need to directly define what should be the adjunction map

$$\mathrm{sk}_m \widetilde{Y}^{(m)} \to Y. \tag{3.3}$$

Well, for $\mu \le m$ the inverse limit $\widetilde{Y}^{(m)}_\mu$ is taken over a category $\mathrm{sk}_m \Delta[\mu]$ with an initial object, namely the identity map on $[\mu]$. Hence, $\widetilde{Y}^{(m)}_\mu = Y_{\mathrm{id}_{[\mu]}} = Y_\mu$, with this identification visibly functorial in $[\mu]$. In other words, we have a natural isomorphism $\mathrm{sk}_m \widetilde{Y}^{(m)} \simeq Y$ induced in degree $\mu \le m$ by the natural projection from the inverse limit $\widetilde{Y}^{(m)}_\mu$ to the object $Y_{\mathrm{id}_{[\mu]}} = Y_\mu$.

Now we will prove that $\widetilde{Y}^{(m)}$ equipped with (3.3) is "final" among simplicial complexes in $C$ equipped with a map from their $m$-skeleton to $Y$. Actually, before doing this we make a minor remark concerning the augmented case. Just as the inverse limit of a finite diagram in a category admitting finite fiber products and products can be realized as a subobject of the product of all of the objects (for much the same reason inverse limits are found inside of products in the category of sets), we could have run through the preceding argument (with $m \ge 0$) in the augmented case as well. The only difference would be that we replace $\Delta[n]$ with $\Delta^+[n]$ and instead of forming the finite inverse limit $\widetilde{Y}^{(m)}_n$ as a subobject of the product of the $Y_\phi$'s, this inverse limit would be a subobject of the fiber product over $Y_{-1}$ of the same $Y_\phi$'s (cut out as a subobject of the product by the "same" relations as in the nonaugmented case).

Of course, we really don't need to replace $\Delta[n]$ with $\Delta^+[n]$ when doing the augmented construction for $n \ge 0$, because the unique map $[-1] \to [n]$ is the initial object of $\Delta^+[n]$ and $Y$ is contravariant (so leaving this one contributing piece out of the inverse limit construction will not affect the limit). Meanwhile, for $n = -1$ we see that $\mathrm{sk}_m \Delta^+[-1]$ is a discrete 1 point category, so forming the limit as suggested above would actually just yield the "constant augmented simplicial object" construction, just as we would want.

Turning to the proof that our above construction really is an $m$-coskeletion in the nonaugmented situation, we note that for a simplicial object $X_\bullet$ in $C$, to give a map $X_\bullet \to \widetilde{Y}^{(m)}$ as simplicial objects is to give maps $X_n \to \widetilde{Y}^{(m)}_n$ naturally in $[n]$. By the very definition of $\widetilde{Y}^{(m)}_n$ as an inverse limit, such a map to $\widetilde{Y}^{(m)}_n$ is equivalent to specifying maps

$$\xi_\phi : X_n \to Y_\phi := Y_k$$

for all $\phi : [k] \to [n]$ with $k \le m$, in a manner which is natural in $[k]$ and in $[n]$. Because of the naturality of $X_\bullet : \Delta \to C$, such naturality for the $\xi_\phi$'s amounts to the requirement that the diagrams

$$\begin{array}{ccc} X_n & \xrightarrow{\xi_\phi} & Y_k \\ {\scriptstyle X_\bullet(\phi)}\downarrow & & \| \\ X_k & \xrightarrow[\xi_{\mathrm{id}_{[k]}}]{} & Y_k \end{array}$$

commute, with $\xi_k := \xi_{\mathrm{id}_{[k]}}$ required to be natural in $[k]$ for $k \le m$. That is, we have to *define* $\xi_\phi := \xi_k \circ X_\bullet(\phi)$ and such a definition does work, thanks to the naturality of $X_\bullet$, provided $\xi_k : X_k \to Y_k$ is natural in $[k]$ for $k \le m$.

This gives a bijection

$$\mathrm{Hom}_{\mathrm{Simp}(C)}(X_\bullet, \widetilde{Y}^{(m)}) \simeq \mathrm{Hom}_{\mathrm{Simp}_m(C)}(\mathrm{sk}_m X_\bullet, Y).$$

Moreover, the construction of this bijection is *exactly* what one gets by first applying the functor $\mathrm{sk}_m$ (i.e., restriction to degrees $k \le m$) and then composing with the map (3.3) defined in degrees $\mu \le m$ in terms of the isomorphism projection from $\widetilde{Y}_\mu^{(m)}$ to $Y_{\mathrm{id}_{[\mu]}} = Y_\mu$.

Hence, we have explicated the existence of both an $m$-coskeleton functor as well as the adjunction morphism $\mathrm{sk}_m \,\mathrm{cosk}_m \to \mathrm{id}$ which turns out to be an isomorphism.

■

We remind the reader that in the augmented case the above proof recovers the earlier explicit construction of $\mathrm{cosk}_0$ in Example 3.3, and we record for later use the following few additional facts concerning the coskeleton functors.

**Corollary 3.10.** *For any $m$-truncated simplicial object $Y$ in $C$ with $m \ge 0$, the natural adjunction map $\mathrm{sk}_m \,\mathrm{cosk}_m Y \to Y$ is an isomorphism. The same is true for augmented objects with $m \ge -1$.*

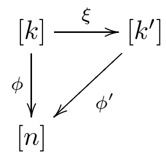
*Moreover, when constructing $(\mathrm{cosk}_m Y)_n$ for $m, n \ge 0$ as an inverse limit over $\mathrm{sk}_m \Delta[n]$ (in either the augmented or non-augmented cases), it suffices to take the limit over the full subcategory of $\mathrm{sk}_m(\Delta[n])$ whose objects $\phi : [k] \to [n]$ are injective set maps with $k \le m$.*

The second part of the corrollary trivially remains true for the augmented case even when we allow $m$ or $n$ to equal $-1$ and use either $\Delta[n]$ or $\Delta^+[n]$ (except that for $n = -1$ we must use $\Delta^+[n]$). The sufficiency of using injective set maps is a key technical observation in Deligne's inductive proof of Theorem 7.9: proper hypercoverings are morphisms of cohomological descent (really see the proof of the ingredient Theorem 7.16).

*Proof.* The isomorphism assertion for the non-augmented case was seen in the construction of $\mathrm{cosk}_m$ for $m \ge 0$, and this also takes care of the augmented case with $m \ge 0$ (e.g., by passage to a slice category). The case $m = -1$ is clear from the explicit construction of $\mathrm{cosk}_{-1}$ as a "constant" comsiplicial object in the augmented case.

As for the second assertion, concerning taking an inverse limit over injective set maps $\phi : [k] \to [n]$ with $k \le m$, the slice category argument reduces the augmented case to the non-augmented case. Thus, we just have to check that the subcategory in $\mathrm{sk}_m \Delta[n]$ consisting of objects $[k] \to [n]$ which are injective maps is "cofinal" in a sufficiently strong sense (roughly, every map $[k] \to [n]$ with $k \le m$ factors through a particularly canonical injective map $[k'] \to [n]$ with $k' \le m$).

In more concrete terms, any $\phi : [k] \to [n]$ with $k \le m$ uniquely factors as $\phi = \alpha \circ \beta$ with $\beta : [k] \to [\overline{k}]$ surjective (so $\overline{k} \le k \le m$) and $\alpha : [\overline{k}] \to [n]$ injective (see (2.2)). Now if we are given a map $\xi$ in $\mathrm{sk}_m(\Delta[n])$ from $\phi$ to some $\phi' : [k'] \to [n]$ with $k' \le m$ (and with corresponding unique injective/surjective factorization $\beta' \circ \alpha'$ through some $[\overline{k}']$), which is to say a commutative diagram




then we trivially see that there is a unique map

$$\overline{\xi} : [\overline{k}] \to [\overline{k}']$$

which is compatible with $\xi$ and the "surjective" parts $\beta$ and $\beta'$ of the factorizations of $\phi$ and $\phi'$. This $\overline{\xi}$ is also compatible with the injections $\alpha : [\overline{k}] \to [n]$ and $\alpha' : [\overline{k}'] \to [n]$. Because of this uniqueness, it is easily checked that $\xi \rightsquigarrow \overline{\xi}$ is compatible with compositions. More importantly, it is clear that *any* factorization of $\phi$ through an *injective* map $\widetilde{\alpha} : [\widetilde{k}] \to [n]$ (say with $\widetilde{k} \le m$) must have the injective $\alpha$ *uniquely* factor through $\widetilde{\alpha}$. Consequently, to compute the inverse limit of all $Y_\phi$'s over $\mathrm{sk}_m \Delta[n]$, it really does suffice to deal with $Y_\alpha$'s for injective $\alpha$.

The point is that $Y$ is a contravariant functor, so in the above notation we have canonical maps $Y_\alpha \to Y_\phi$ defined by $Y(\beta)$, and the composition compatibility of $\xi \rightsquigarrow \overline{\xi}$ and the above unique factorization of $\alpha$ through any other injective $\widetilde{\alpha}$'s guarantees well-definedness and naturality (in $\phi$) of defining maps to $Y_\phi$

via factorization of $\phi$ through any injective map $[\kappa] \to [n]$ with $\kappa \le m$. This concludes the proof that for computing $(\operatorname{cosk}_m Y)_n$ it suffices to take the inverse limit only over injective maps $[k] \to [n]$ (with $k \le m$). ■

Although the adjunction $\mathrm{id} \to \operatorname{cosk}_m \operatorname{sk}_m$ is generally not an isomorphism (and in fact this sort of map will play an important role in the definition of hypercovers), there is a class of objects on which this adjunction does induce an isomorphism: $n$-coskeleta for $n \le m$.

**Corollary 3.11.** *For $0 \le n \le m$, the natural map*

$$\rho_{m,n} : \operatorname{cosk}_n \to \operatorname{cosk}_m \operatorname{sk}_m \operatorname{cosk}_n \tag{3.4}$$

*of functors on $\mathrm{Simp}_n(C)$ is an isomorphism. The same holds on $\mathrm{Simp}_n^+(C)$ allowing $-1 \le n \le m$.*

Before giving the proof, we make some remarks. By taking $n = 0$ in the augmented case, this corollary "explains" the mechanical nature of the Čech construction. Namely, given a covering map $Y_0 \to Y_{-1}$, the truncation $Y_{\le m}$ in degrees $\le m$ on the Čech construction $\operatorname{cosk}_0(Y_0/Y_{-1})$ has $\operatorname{cosk}_m(Y_{\le m})$ canonically isomorphic to $\operatorname{cosk}_0(Y_0/Y_{-1})$, so we are back where we began. In other words, one can either view the Čech construction as being a one-step application of $\operatorname{cosk}_0$ or alternatively as a process of applying $\operatorname{cosk}_n$ and $\operatorname{sk}_{n+1}$ to build the $(n+1)$th level, then $\operatorname{cosk}_{n+1}$ and $\operatorname{sk}_{n+2}$ to build the $(n+2)$th level, and so on (always secretly returning back to the original $\operatorname{cosk}_0$ after each coskeleton step).

The version of Corollary 3.11 which is stated in [SGA4, Exp Vbis, 7.1.2] is a little sloppy: the inequality on subscripts is backwards and slightly more dangerously the intermediate functor $\operatorname{sk}_m$ is omitted (by "abuse of notation"). However, within the framework of [SGA4] such abuse of notation is permissible because they also discuss a left adjoint to $\operatorname{sk}_m$ which we have not addressed. Thus, we will have to argue a little less cleverly. If we were sloppy and omitted the $\operatorname{sk}_m$, we would be tempted to just use adjointness to reduce to the obvious transivity

$$\operatorname{sk}_n^m \circ \operatorname{sk}_m \simeq \operatorname{sk}_n$$

of truncation functors (where $\operatorname{sk}_n^m : \mathrm{Simp}_m(C) \to \mathrm{Simp}_n(C)$ is the evident "restriction"). The more refined approach in [SGA4] permits such an argument, but from the way we've set things up we need to do a little more work (on the other hand, being forced to work a little more directly with the definitions and constructions is a good way to get used to new concepts).

*Proof.* The augmented cases $m = -1$, as well as $n = -1$ with $m$ arbitrary, are clear "by hand", since the coskeleton of a "constant" $n$-truncated (augmented) simplicial object is clearly the associated constant (augmented) simplicial object. Thus, we may assume $m \ge n \ge 0$. By passing to slice categories we may also reduce to the non-augmented situation.

Now fix an $n$-truncated simplicial object $Y$, and consider a map

$$f : X \to \operatorname{cosk}_m \operatorname{sk}_m \operatorname{cosk}_n Y.$$

We must show that there is a unique map $g : X \to \operatorname{cosk}_n Y$ whose composite with $\rho_{m,n}(Y)$ in (3.4) is $f$ (and then by Yoneda we'll be done). To check the equality of two maps to an $m$-coskeleton, it suffices to check equality on the level of $m$-truncations. In other words, given a map $f' : \operatorname{sk}_m X \to \operatorname{sk}_m \operatorname{cosk}_n Y$, we must show that there is a unique map $g : X \to \operatorname{cosk}_n Y$ such that $\operatorname{sk}_m(g) = f'$. Well, to give $g$ is to give its restriction $g' : \operatorname{sk}_n X \to Y$ on $n$-skeleta (here we are using that $\operatorname{sk}_n \operatorname{cosk}_n$ is isomorphic to the identity via adjunction), and $g'$ has no choice but to be the restriction of $\operatorname{sk}_m(g)$ to $\operatorname{sk}_n(X)$. Thus, to find $g$ with $\operatorname{sk}_m(g) = f'$ the only possibility is to take $g'$ to be the restriction of $f'$ to $n$-skeleta, so we must show that for any $f' : \operatorname{sk}_m X \to \operatorname{sk}_m \operatorname{cosk}_n Y$, the $n$-skelecton restriction

$$\operatorname{sk}_n^m(f') : \operatorname{sk}_n X \to \operatorname{sk}_n^m \operatorname{sk}_m \operatorname{cosk}_n Y \simeq \operatorname{sk}_n \operatorname{cosk}_n Y \simeq Y$$

corresponds under adjointness to a map $F : X \to \operatorname{cosk}_n Y$ whose $m$-skeleton restriction is $f'$.

In order to prove that $f' = \operatorname{sk}_m(F)$, we just have to show that two maps

$$h, h' : \operatorname{sk}_m X \to \operatorname{sk}_m \operatorname{cosk}_n Y$$

which coincide on $n$-skeleta must be equal, when $0 \leq n \leq m$. If $m = n$, there's nothing to do. In general, with $m > n$ fixed we may induct on $m$ and so we may suppose we have equality on $\mathrm{sk}_{m-1}$'s. The only problem is to show that the two maps

$$h_m, h'_m : X_m \rightrightarrows (\mathrm{cosk}_n\, Y)_m = \varprojlim_{\mathrm{sk}_n \Delta[m]} Y_\phi \tag{3.5}$$

coincide (where the inverse limit is taken over maps $\phi : [k] \to [m]$ with $k \leq n$). It suffices to consider equality after composing with projection to each $Y_\phi = Y_k$. From the very construction of $\mathrm{cosk}_n\, Y$ as a simplicial object, this projection map to $Y_\phi$ is canonically identified with

$$(\mathrm{cosk}_n\, Y)(\phi) : (\mathrm{cosk}_n\, Y)_m \to (\mathrm{cosk}_n\, Y)_k \simeq Y_k$$

via the canonical identification of $(\mathrm{cosk}_n\, Y)_k$ with $Y_k$ for $k \leq n$. But by *naturality* our maps $h$ and $h'$ with degree $m$ restriction (3.5) must respect functoriality with respect to $\phi$, so checking that the composites of (3.5) with projection to $Y_\phi = Y_k$ yield a common map is equivalent to checking that the composition of

$$X(\phi) : X_m \to X_k$$

with $h_k$ and $h'_k$ yields a common map. But due to our assumption that $h$ and $h'$ have the same restrictions to $n$-skeleta, the fact that $k \leq n$ implies that we're done. ■

We end this secion with a useful consequence of Corollary 3.11. For $0 \leq n \leq m$ we have a functor

$$\mathrm{sk}_n^m : \mathrm{Simp}_n(C) \to \mathrm{Simp}_m(C)$$

which is just "restriction". We can ask if this has a right adjoint $\mathrm{cosk}_m^n$, an analogue of the coskeleton in the context of truncated objects (now depending on $n$ and $m$). A natural guess is to use $\mathrm{sk}_m \mathrm{cosk}_n$, with

$$\mathrm{sk}_n^m \circ (\mathrm{sk}_m \mathrm{cosk}_n) = \mathrm{sk}_n \circ \mathrm{cosk}_n = \mathrm{id}$$

as the adjunction. Indeed, this works:

**Corollary 3.12.** *For $0 \leq n \leq m$, the functor $\mathrm{sk}_m \mathrm{cosk}_n$ is right adjoint to $\mathrm{sk}_n^m$ via the above indicated adjunction isomorphism. The same holds in the augmented case with $-1 \leq n \leq m$.*

*Proof.* We will work out a bijection on the level of Hom-sets, and leave it to the reader to see that the adjunction morphism is as expected. For $m$-truncated $X$ and $n$-truncated $Y$, we have the following calculation (dropping the category labels on Hom-sets so as to treat augmented and nonaugmented cases at the same time):

$$\mathrm{Hom}(\mathrm{sk}_n^m X, Y) = \mathrm{Hom}(\mathrm{sk}_n^m \mathrm{sk}_m \mathrm{cosk}_m X, Y) = \mathrm{Hom}(\mathrm{sk}_n \mathrm{cosk}_m X, Y) = \mathrm{Hom}(\mathrm{cosk}_m X, \mathrm{cosk}_n Y).$$

Now by Corollary 3.11 we have

$$\mathrm{cosk}_n Y \simeq \mathrm{cosk}_m \mathrm{sk}_m \mathrm{cosk}_n Y,$$

so

$$\begin{aligned} \mathrm{Hom}(\mathrm{sk}_n^m X, Y) &= \mathrm{Hom}(\mathrm{cosk}_m X, \mathrm{cosk}_m \mathrm{sk}_m \mathrm{cosk}_n Y) \\ &= \mathrm{Hom}(\mathrm{sk}_m \mathrm{cosk}_m X, \mathrm{sk}_m \mathrm{cosk}_n Y) \\ &= \mathrm{Hom}(X, \mathrm{sk}_m \mathrm{cosk}_n Y). \end{aligned}$$

This provides the desired right adjoint functor. ■

For technical reasons in Deligne's proofs, it is convenient to discuss a mild extension of the simplicial theory to the multisimplicial case (we will only need the bisimplicial case). For categories $C_1, \ldots, C_r$, we can form the product category $C_1 \times \cdots \times C_r$ whose objects are $r$-tuples $(c_1, \ldots, c_r)$ with $c_j$ an object of $C_j$ and whose morphisms from $(c_i)$ to $(c'_i)$ are $r$-tuples $(f_1, \ldots, f_r)$ with $f_i : c_i \to c'_i$ (and composition defined in the evident manner). For $r \geq 1$, consider the $r$-fold product category $\Delta^r$, with objects simply $r$-tuples

$([n_1], \dots, [n_r])$ with $n_1, \dots, n_r$ non-negative integers, etc. We also define $\Delta^0$ to denote the category consisting of a single discrete point.

**Definition 3.13.** A *multisimplicial object* (or *multisimplicial complex*) in a category $C$ is a contravariant functor $X : \Delta^r \to C$ for some $r \geq 0$. We say such $X$ is *r-multisimplicial*.

We define the cosimplicial variant by using covariant functors, and for fixed $r$ both concepts form categories $r\,\mathrm{Simp}(C)$ and $r\,\mathrm{Cosimp}(C)$ in an obvious manner.

This is a multi-dimensional version of (2.3). As an example, a bisimplicial object is a very complicated first quadrant diagram with lots of little commutative squares. Just look at the diagram of arrows in (7.4), ignoring the left and bottom edges. In concrete terms, this picture is an illustration of the typical way one encounters a bisimplicial object: when our category $C$ admits products, then when given two simplicial objects $X_\bullet$ and $Y_\bullet$ we can form the products $X_p \times Y_q$ and endow the resulting 2-dimensional grid with a bisimplicial structure coming from products of the maps among the $X_p$'s and the maps among the $Y_q$'s. Definition 3.13 is a more efficient way to package the idea than to try to write down the oodles of relations explicitly. We ignore the issue of defining truncated subcategories and augmented variants (in particular, we don't contemplate coskeleton functors for multisimplicial objects).

Of course, the cases $r = 1, 2$ are the most interesting ones. Note that in the entire preceding development we only ever imposed conditions on $C$ which were *inherited* by $\mathrm{Simp}(C)$, so the following trivial lemma is sometimes a useful trick.

**Lemma 3.14.** *For any category $C$, there is a natural equivalance* (*in fact, there are many!*)

$$r\,\mathrm{Simp}(C) \simeq \mathrm{Simp}((r-1)\,\mathrm{Simp}(C))$$

*for all $r \leq 1$.*

Of course, since the equivalence in this lemma is by no means special (there are lots of ways to make an equivalence), one certainly doesn't want to use this lemma to make recursive *definitions* in the multisimplicial world (as then one would constantly have to check it didn't matter which equivalence one chose). On the other hand, when doing proofs involving previously-defined concepts, this lemma is sometimes a handy device.

*Proof.* We may assume $r > 1$. Since $\Delta^r \simeq \Delta \times (\Delta^{r-1})$ (in many ways; pick one), this lemma comes down to the claim that for small categories $D$ and $D'$, to give a contravariant functor $D \times D' \to C$ is really to assign to each object $d$ in $D$ a contravariant functor $X_d : D' \to C$ such that the assignment $d \rightsquigarrow X_d$ is itself contravariant (to each map $d \to \delta$ we get a natural transformation $X_\delta \to X_d$ such that ...). But this claim is obvious. ∎

## 4. Hypercovers

Now that we have constructed coskeleta, we are ready to apply them to define the concept of hypercover. Rather crudely, the idea is to begin with a covering $\{U_i\}$ of a topological space $X$, then at the next level we pick coverings $\{V_{ij}\}$ of the $U_i$'s, then at the next level we pick coverings $\{W_{ijk}\}$ of the $V_{ij}$'s and so on, with the $(n+1)$th stage arising as a "cover" of everything from stages in degrees $\leq n$. The precise way of formulating this latter condition is in terms of the degree $n+1$ part of the functor $\mathrm{cosk}_n$:

**Definition 4.1.** Let $C$ be a category admitting finite products and finite fiber products (i.e., admitting finite inverse limits). Let $\mathbf{P}$ be a class of morphisms in $C$ which is stable under base change, preserved under composition (hence under products), and contains all isomorphisms. A simplicial object $X_\bullet$ in $C$ is said to be a $\mathbf{P}$*-hypercovering* if, for all $n \geq 0$, the natural adjunction map

$$X_\bullet \to \mathrm{cosk}_n(\mathrm{sk}_n(X_\bullet)) \tag{4.1}$$

induces a map $X_{n+1} \to (\mathrm{cosk}_n(\mathrm{sk}_n(X_\bullet)))_{n+1}$ in degree $n+1$ which is in $\mathbf{P}$. If $X_\bullet$ is an augmented simplicial complex, we make a similar definition but also require the case $n = -1$ (and we then say $X_\bullet$ is a $\mathbf{P}$-hypercovering of $X_{-1}$).

By Theorem 7.5 below, the class of morphisms which are universally of cohomological descent (see Definition 6.5) can be taken for **P** in the preceding definition.

*Example* 4.2. When $C$ is a suitable category of spaces and **P** is the class of proper surjective maps, we speak of a *proper hypercovering* (supressing explicit mention of the surjectivity condition), and likewise when **P** is the class of étale surjective maps, we speak of *étale hypercoverings.*

It seems that the only interesting case of Definition 4.1 is the augmented case, but the augmentation isn't really needed for all proofs (though it is needed for the interesting ones), and (by means of slice categories) we can often reduce the augmented case to the nonaugmented case. Thus, for reasons of both technical simplicity as well as clearer generality we will keep in mind both the augmented and nonaugmented cases of this definition (e.g., one must beware that an augmented simplicial object which is a hypercovering need not have its underlying non-augmented incarnation a hypercovering as well; this usually is very false, due to the failure of coskeletons to commute with "forgetting the augmentation").

One might ask for a definition of when a map

$$u_\bullet : X_\bullet \to Y_\bullet$$

between augmented simplicial objects (with a common $S$ in degree $-1$) is to be regarded as a **P**-hypercovering (recovering Definition 4.1 when $Y_\bullet/S$ is constant). This issue will be addressed at the end of §7, after the proof of Theorem 7.17.

*Example* 4.3. If $C$ is a category with finite inverse limits and a Grothendieck topology (e.g., the category of topological spaces, or schemes, or schemes with the étale topology), it is natural to take **P** to consist of the class of covering maps in the topology. This example will yield Čech theory as a special case. The *amazing* fact is that one can actually sometimes "compute" cohomology using **P**'s that are rather *unlike* covering maps for a Grothendieck topology, such as the class of proper surjections for either topological spaces or schemes with the étale topology.

*Example* 4.4. Let's take $C$ to be a category of spaces, and **P** the class of surjective étale morphisms. If we let $X$ be a space and $\{U_i\}$ a covering of $X$, then for $U = \coprod U_i$ we have that $X_\bullet = \operatorname{cosk}_0(U/X)$ is an (augmented) **P**-hypercovering. Indeed, for every $n \geq 0$ the natural map

$$X_\bullet \to \operatorname{cosk}_n(\operatorname{sk}_n X_\bullet)$$

is an isomorphism by Corollary 3.11, hence induces an isomorphism in degree $n+1$. Meanwhile, for $n = -1$ this map induces the augmentation map $U \to X$ in degree 0, which by construction is a surjective local isomorphism. This same argument shows that in the categorical generality of Definition 4.1, the augmented simplicial object $\operatorname{cosk}_0(S'/S) \to S$ is a **P**-hypercovering if and only if $S' \to S$ is a morphism of type **P**.

We'll see in Corollary 4.14 that in the category of topological spaces or schemes, with **P** either the proper surjections or the étale surjections, then the face and degeneracy maps of an *augmented* **P**-hypercovering are necessarily proper or étale respectively.

*Example* 4.5. We can somewhat generalize the preceding example by means of the concept of an *m-truncated* **P***-hypercovering* $X_\bullet$ for $m \geq 0$. This concept is defined as follows. For $n < m$, we require that the adjunction map of $m$-truncated objects

$$X_\bullet \to \operatorname{cosk}_n^m \operatorname{sk}_n^m X_\bullet \tag{4.2}$$

be of class **P**, where the $m$-truncated coskeleton functor $\operatorname{cosk}_n^m$ is as in Corollary 3.12. When $m = 0$ and we work with augmented objects we see that an $m$-truncated **P**-hypercovering is just an augmentation map $X_0 \to X_{-1}$ which is of class **P**. Also, it is clear (check!) that $\operatorname{sk}_m$ carries **P**-hypercovers to $m$-truncated **P**-hypercovers (and ditto for $\operatorname{sk}_m^{m'}$ applied to $m'$-truncated **P**-hypercovers, where $m' \geq m$).

In order to generalize Example 4.4, suppose we are given an $m$-truncated **P**-hypercovering $Y_\bullet$ with some $m \geq 0$ (either in the augmented or non-augmented sense). Then $X_\bullet = \operatorname{cosk}_m(Y_\bullet)$ is a simplicial object (augmented when $Y_\bullet$ is), and we claim it is automatically a **P**-hypercovering. For $m = 0$ and $Y_\bullet$ augmented, this really is Example 4.4.

To see (for any $m \geq 0$ above) that the $m$-coskeleton $X_\bullet$ of an $m$-truncated **P**-hypercovering is itself a **P**-hypercovering, we must show that for each $n$, the adjunction $X_\bullet \to \operatorname{cosk}_n \operatorname{sk}_n X_\bullet$ induces a map of class

$\mathbf{P}$ in degree $n+1$. When $n < m$, we can compute in degree $n+1 \le m$ by first applying $\mathrm{sk}_m$. That is, we consider

$$\mathrm{sk}_m X_\bullet \to \mathrm{sk}_m \operatorname{cosk}_n \mathrm{sk}_n X_\bullet \tag{4.3}$$

in degree $n+1 \le m$. But $\mathrm{sk}_m X_\bullet = Y_\bullet$ (due to the definition of $X_\bullet$), so $\mathrm{sk}_n X_\bullet = \mathrm{sk}_n^m Y_\bullet$. Since $\mathrm{sk}_m \operatorname{cosk}_n = \operatorname{cosk}_n^m$ by Corollary 3.12, the map (4.3) in degree $n+1 \le m$ is identified with the degree $n+1$ level of

$$Y_\bullet \to \operatorname{cosk}_n^m \mathrm{sk}_n^m Y_\bullet.$$

However, this latter map is readily checked to be the canonical adjunction (check!), so since $Y_\bullet$ is *by hypothesis* an $m$-truncated $\mathbf{P}$-hypercovering we conclude that our map (4.2) in degree $n+1 \le m$ is of class $\mathbf{P}$.

Now consider what happens in degree $n+1$ when $n \ge m$. In this case, since $X_\bullet$ is an $m$-skeleton, we may apply Corollary 3.11 (with the roles of the variables $n$ and $m$ reversed) to conclude that the adjunction map

$$X_\bullet \to \operatorname{cosk}_n^m \mathrm{sk}_n^m X_\bullet$$

is actually an isomorphism and hence in degree $n+1$ is certainly of class $\mathbf{P}$!

Assuming that $C$ admits finite products and finite fiber products, we can form an ad hoc product and fiber product for simplicial objects of $C$, by forming the products in each separate degree. It is trivial to check that this ad hoc construction serves as such a product within the category $\mathrm{Simp}(C)$ (and the same goes through in $\mathrm{Simp}^+(C)$). More generally, again working degree-by-degree, we see that $\mathrm{Simp}(C)$ and $\mathrm{Simp}^+(C)$ admit finite inverse limits, and similarly for categories of $n$-truncated objects, with the functors $\mathrm{sk}_n$ and $\mathrm{sk}_n^m$ (for $n \le m$) commuting with formation of such limits. Since $\operatorname{cosk}_n$ and $\operatorname{cosk}_n^m$ (for $n \le m$) are right adjoints, they tautologically commute with formation of finite inverse limits (such as finite products or fiber products). Consequently, by looking back at the definitions of hypercovers and $m$-truncated hypercovers, we arrive at:

**Lemma 4.6.** *Assume $C$ admits finite inverse limits. The product of two $\mathbf{P}$-hypercovers is a $\mathbf{P}$-hypercover. If the class $\mathbf{P}$ is preserved under formation of fiber products of morphisms (i.e., $f \times_g h$ is of type $\mathbf{P}$ if $f, g, h$ are), then a fiber product of $\mathbf{P}$-hypercovers is a $\mathbf{P}$-hypercover. In particular, any two augmented $\mathbf{P}$-hypercovers of a common object $S$ in degree $-1$ admit a common "refinement".*

*This is also all true for the $m$-truncated case with any $m$.*

The hypothesis on the stability of $\mathbf{P}$ under fiber products is satisfied in most (all?) interesting cases.

*Proof.* The preceding discussion gives us a handle on the behavior of the adjunction morphisms (as in the definition of hypercoverings) under products and fiber products of simplicial objects in $C$, so the problem comes down to showing that in the category $C$, a product of two maps of class $\mathbf{P}$ is of class $\mathbf{P}$ (the fiber product case being immediate from the hypotheses in that case). Since $\mathbf{P}$ is preserved under composition and base change and contains isomorphisms, the preservation of property $\mathbf{P}$ under products is clear. ■

Now it is time to show that there are lots of hypercoverings aside from the ones coming from Čech theory. We are particularly interested in the case when $C$ is the category of schemes (given the étale topology) or when $C$ is the category of topological spaces. In these two respective cases we may take $\mathbf{P}$ to be the class of proper surjective maps, and we refer to $\mathbf{P}$-hypercoverings as *proper hypercoverings*. We'll see in Corollary 4.14 that all of the face and degeneracy maps for an *augmented* proper hypercovering are necessarily proper, with the degeneracies then necessarily closed immersions (as each is the section to some face map). The key fact for our purposes is the following, which we will prove later (as Theorem 4.16):

**Theorem 4.7.** *Let $S$ be a separated scheme of finite type over a field $k$. Then there exists a dense open immersion $S \hookrightarrow \overline{S}$ into a proper $k$-scheme and an augmented proper hypercovering $\overline{X}_\bullet$ of $\overline{S}$ such that each $\overline{X}_n$ is a regular $k$-scheme (so each $\overline{X}_n$ is $k$-smooth for perfect $k$) and the part $D_n$ of $\overline{X}_n$ over $\overline{S} - S$ is a strict normal crossings divisor in $\overline{X}_n$.*

*In particular, $\overline{X}_\bullet - D_\bullet$ is a proper hypercovering of $S$ by regular algebraic $k$-schemes, and when $k = \mathbf{C}$ the topological space $S(\mathbf{C})$ admits a topological proper hypercovering $X_\bullet(\mathbf{C})$ with each $X_n(\mathbf{C})$ a Hausdorff complex manifold.*

We note that, just as with the Čech example in Example 4.4, the $X_n$'s in Theorem 4.7 are typically very disconnected. In order to prove Theorem 4.7, we will of course need to use resolution of singularities (in the form given by deJong if $k$ has positive characteristic). In some sense, this plays the role of the inductive step. However, to get things off the ground, we first need to develop a few more tools for *constructing* new hypercovers from old ones. For example, we still have yet to discuss how to "refine" a hypercovering in higher degrees without affecting lower degrees. After we spend some time developing these additional tools, we will prove Theorem 4.7.

Let's begin by trying to make a naive proof of Theorem 4.7. By resolution of singularities, for $S$ separated of finite type over $k$ there exists a map $X_0 \to S$ of class $\mathbf{P}$ (i.e., a proper surjection) with $X_0$ regular. The Čech approach (i.e., $\mathrm{cosk}_0$) would introduce $X_0 \times_S X_0$ at the next level, but of course this is rarely again regular. Hence, we apply resolution of singularities to get a proper surjection

$$X_1' \twoheadrightarrow X_0 \times_S X_0$$

with $X_1'$ regular. We have two candidate face maps $X_1' \to X_0$, but there's no section to serve as a degeneracy map (e.g., the diagonal for $X_{0/S}$ can't help anymore). But we can just define

$$X_1 = X_1' \coprod X_0$$

to force a section, and in this way we have built a 1-truncated augmented simplicial object $X_{\leq 1}$ of the desired type: the map

$$X_{\leq 1} \to \mathrm{cosk}_0\, \mathrm{sk}_0\, X_{\leq 1}$$

in degree 1 is

$$X_1' \coprod X_0 = X_1 \to X_0 \times_S X_0,$$

given by the proper surjective resolution map on $X_1'$ and the diagonal map on $X_0$. This is a proper surjection (since $X_0$ is $S$-separated). Passing to the construction in degree 2 is going to be a bit more complicated to do by hand because there are many more relations to deal with, and computing the coskeleton property as required to be a hypercovering is going to be a real mess if we work explicitly.

The basic strategy is that once we have an $n$-truncated solution $X_{\leq n}$, we apply $\mathrm{cosk}_n$ (which has no impact in degrees $\leq n$) and apply resolution of singularities to the inverse limit beast that we get in degree $n+1$, and then define $X_{n+1}$ to be a disjoint union of this resolution with several copies of $X_n$ (so as to have degeneracy maps from degree $n$ to degree $n+1$). But as we have mentioned, it will be somewhat unpleasant to directly compute whether our $(n+1)$-truncated construction is actually a truncated hypercovering (there is now a condition to be checked in degree $n+1$). Thus, in order to make the argument go more smoothly, it behooves us to develop some general tools for manipulating truncated hypercovers.

But first, lets look at an important class of examples for which the 0-coskeleton already works even when $S$ is not regular.

*Example* 4.8. Let $Z$ be a regular noetherian scheme or compact complex analytic space, and let $D$ in $Z$ be a strict normal crossings divisor. By this we mean that $D$ is a (reduced) union of distinct irreducible divisors $D_1, \dots, D_n$ in $Z$ with each $D_j$ regular and all higher order overlaps among the $D_j$'s being regular (of the expected codimension). Let $X_0 = \coprod D_j$. In this case, $X_0 \to D$ is a proper surjection, so

$$\mathrm{cosk}_0(X_0/D) \to D$$

is a proper hypercovering. The constituent terms in each degree are already regular, since in degree $p \geq 0$ we just get the $(p+1)$-fold overlaps in $D$ among the $D_j$'s (with repetitions allowed), all of which are regular of codimension $\leq p+1$ in $Z$ (with equality exactly for the overlaps of $p+1$ distinct $D_j$'s). If we consider just those terms in degree $p$ of codimension exactly $p+1$, we get a finite collection of objects (sort of a geometric counterpart to the "alternating subcomplex" when doing "unordered" Čech theory). We'll return to this example in Example 7.8.

The starting point for the conceptual approach to construction questions for hypercoverings is the following definition which extends the Dold-Kan construction to non-abelian situations. We return therefore to the original general setup of an arbitrary category $C$.

**Definition 4.9.** Assume that our category $C$ admits finite coproducts (disjoint unions in geometric cases, ordinary direct sums in abelian categories). We say that a simplicial object $X_\bullet$ in $C$ is *split* if there exist subobjects $NX_j$ in each $X_j$ such that the natural map

$$\coprod_{\phi:[n]\twoheadrightarrow[m],m\leq n} NX_\phi \to X_n$$

is an isomorphism for every $n \geq 0$, where $NX_\phi := NX_m$ for a surjection $\phi : [n] \twoheadrightarrow [m]$ and the map $NX_\phi \to X_n$ is the inclusion $NX_\phi \hookrightarrow X_m$ followed by $X(\phi)$. We call the specification of such subobjects $NX_j$ a *splitting* of $X_\bullet$.

For $m$-truncated objects (with $m \geq 0$) or augmented objects or $m$-truncated augmented objects we make a similar definition (with a fixed augmented object in degree $-1$).

The proof of the Dold-Kan correspondence shows that if $C$ is an abelian category, then *every* simplicial object in $C$ is *canonically* split (and likewise in the augmented and truncated situations). Let's see to what extent the subobjects $NX_j$ in Definition 4.9 are uniquely determined for a split object $X_\bullet$. Obviously we must have $NX_0 = X_0$, and in the abelian category case there is some flexibility in higher degrees (due to the non-uniqueness of "complementary factors" for a direct summand of a module). I claim that in "geometric" situations, the other $NX_j$'s are uniquely determined. To make this precise, we first make a definition.

**Definition 4.10.** Let $C$ be an arbitrary category admitting finite coproducts. We say that $C$ *admits unique complements* if for any object $Z$ with subobject $X$ and any isomorphism

$$X \coprod Y \simeq Z \simeq X \coprod Y'$$

in $C$ lifting the identity on $X$, this map arises from a unique isomorphism $Y \simeq Y'$. We then call the uniquely determined $Y$ the *complement* to $X$ in $Z$.

This definition applies when $C$ is the category of topological spaces, since if $X = U \coprod V$ is a coproduct (i.e., disjoint union) in $C$ then the only decomposition $X = U \coprod V'$ is with $V' = V$. The same applies to schemes, or sheaves of sets (but not to abelian sheaves on $X$ or more generally abelian categories, though in that context Dold-Kan gives a canonical splitting for the categories of (co)simplicial objects).

**Lemma 4.11.** *Let $C$ be a category admitting finite coproducts, with unique complements. Any split simplicial object $X_\bullet$ in $C$ has its subobjects $NX_j$ uniquely determined up to unique isomorphism. The same is true in the augmented and truncated situations.*

*Proof.* We argue by induction, the case of degree 0 being clear, so we may pick $n > 0$ and assume the result is known in degrees $< n$. Thus, we have unique $NX_j$'s for $0 \leq j < n$ and for any $m \leq n-1$ we have $X_{n-1}$ is a coproduct of $NX_\phi$'s for surjective $\phi : [n-1] \twoheadrightarrow [m]$ (so $NX_\phi = NX_m$). If we consider the degeneracy map $s^j_{n-1} : X_{n-1} \to X_n$ for $0 \leq j \leq n-1$, then the component $NX_m = NX_\phi$ of $X_{n-1}$ labelled by $\phi$ is carried by $s^j_{n-1}$ to the component $NX_m$ of $X_n$ labelled by the surjection $\phi \circ \sigma^j_{n-1}$. Due to the uniqueness of the factorization of surjections in $\Delta$ via (2.2), the surjective map $\phi \circ \sigma^j_{n-1}$ uniquely determines $j$ and $\phi$, whence the canonical map

$$\coprod_{0\leq j\leq n-1} s^j_{n-1} : \coprod_{0\leq j\leq n-1} X_{n-1} \to X_n$$

is a direct factor because $X_\bullet$ *is* actually split, and $NX_n$ has no choice but to be the unique complement to this direct factor (the existence of which is ensured by the hypothesis that $X_\bullet$ is split).

■

*From now on in this section, we assume $C$ admits finite inverse limits and finite coproducts, as well as unique complements.* Our hypotheses ensures that we can make sense of coskeleta and hypercoverings in $C$, and split simplicial objects in $C$ have unique splittings. Moreover, the preceding proof makes it clear that the splitting (i.e., formation of $NX_j$'s for a split object) is functorial. For a split object $X$, we will call the $NX_j$'s the (*split*) *components* of $X$.

In order to make inductive constructions, it is convenient to give a precise mechanism by which a split simplicial object $X_\bullet$ in $C$ can be reconstructed from the "bare" objects $NX_j$. We wish to regard the notion of a split simplicial object as a "derived" version of the notion of coproduct: to give a map from a split object $X_\bullet$ to an arbitrary simplicial object we ought to just need to say what happens on the $NX_j$'s.

To make this precise, we will formulate things on the level of truncated objects, describing how to functorially reconstruct a split $(n+1)$-truncated $X$ from the data of the $n$-split $Y := \mathrm{sk}_n X$ and the complementary object $N := NX_{n+1}$. Actually, there is one more piece of information we need, intuitively corresponding to the "gluing data": from the structure $X_\bullet$ we get a natural composite map

$$\beta : N \hookrightarrow X_{n+1} \to (\mathrm{cosk}_n \, \mathrm{sk}_n \, X)_{n+1} = (\mathrm{cosk}_n \, Y)_{n+1}.$$

It is the triple of data $\alpha(X) = (Y, N, \beta)$ which will suffice to reconstruct our $(n+1)$-truncated split object $X$. It is clear how to define a category of triples (consisting of an $n$-truncated simplicial object in $C$, an object of $C$, and a suitable map involving the degree $n+1$ part of an $n$-coskeleton) in which $\alpha(X)$ lives.

To make things more explicit, notice that if $Z$ is any $(n+1)$-truncated simplicial object (split or not) and if we are given a map $f : X \to Z$, then we get a map

$$f' = \mathrm{sk}_n(f) : Y = \mathrm{sk}_n(X) \to \mathrm{sk}_n(Z)$$

and a map

$$f'' : N = NX_{n+1} \longrightarrow X_{n+1} \xrightarrow{f_{n+1}} Z_{n+1}$$

such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\beta} & (\mathrm{cosk}_n \, Y)_{n+1} \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle (\mathrm{cosk}_n(f'))_{n+1}} \\ Z_{n+1} & \longrightarrow & (\mathrm{cosk}_n \, \mathrm{sk}_n^{n+1} \, Z)_{n+1} \end{array}$$

commutes, thanks to the naturality of the degree $n+1$ map induced by the adjunction $\mathrm{id} \to \mathrm{cosk}_n^{n+1} \, \mathrm{sk}_n^{n+1}$ (applied to the morphism $f$). Note that from a slightly more intrinsic "truncated" point of view, we could have replaced $\mathrm{cosk}_n \, \mathrm{sk}_n^{n+1}$ with $\mathrm{cosk}_n^{n+1} \, \mathrm{sk}_n^{n+1}$ in the above digram without harming anything.

The above considerations lead us to:

**Theorem 4.12.** *With the category $C$ as above, fix an integer $n \geq 0$.*

(1) *For any split $n$-truncated simplicial object $Y$ in $C$, any object $N$ in $C$, and any morphism*

$$\beta : N \to (\mathrm{cosk}_n \, Y)_{n+1},$$

*there exists a split $(n+1)$-truncated simplicial object $X$ in $C$ with $\alpha(X) \simeq (Y, N, \beta)$.*

(2) *This $X$ is unique up to unique isomorphism in the sense that for any $(n+1)$-truncated simplicial $Z$ in $C$, the natural map*

$$\mathrm{Hom}_{\mathrm{Simp}_{n+1}(C)}(X, Z) \to \mathrm{Hom}(\alpha(X), (\mathrm{sk}_n^{n+1} \, Z, Z_{n+1}, Z_{n+1} \to (\mathrm{cosk}_n^{n+1} \, \mathrm{sk}_n^{n+1} \, Z)_{n+1}))$$

*is bijective.*

*This all remains true in the augmented case too (still with $n \geq 0$).*

*Proof.* The augmented case trivially reduces to the non-augmented case by the usual slice argument, so now consider the non-augmented case. On the level of objects there is not much mystery for the first part concerning how to define $X_{n+1}$: we take a disjoint union of $N$ with a lot of copies of $NY_m$'s indexed by surjections $[n+1] \to [m]$ over all $m \leq n$. The tedious part is to define the face and degeneracy maps correctly. The gory details (actually, they're not really all that gory) are given in [SGA4, Exp Vbis, pp.66-7] in a very readable manner (with Lemma 2.2 helping to keep things under control). ∎

The upshot of all of this is that in order to "promote" a split $n$-truncated object $Y$ to a split $(n+1)$-truncated object $X$ (we want $\mathrm{sk}_n \, X = Y$), we just have to specify a map

$$\beta : N \to (\mathrm{cosk}_n \, Y)_{n+1}$$

which tells us how to "glue an $(n+1)$-cell" $N$ in at degree $n+1$. The term $X_{n+1}$ will be a coproduct of $N$ and various split components of $Y$, but the face and degeneracy maps are a little more complicated to write down explicitly (this is where $\beta$ enters, and is the reason we referred to [SGA4] for the tedious details).

Now we are in position to prove the key theorem which will enable us to prove Theorem 4.7. Instead of maintaining the level of categorical generality that we have using throughout, for the key theorem we will specialize ourselves to one of three situations:

- $C$ is the category of spaces over a fixed base, with $\mathbf{P}$ the class of proper surjections;
- $C$ is the category of spaces "étale" over a single space (i.e., structure map is étale), with $\mathbf{P}$ the class of surjective étale maps;
- $C$ is any Grothendieck topos, $\mathbf{P}$ is the class of epic morphisms in the topos.

Notice that in each case, if $S$ is an object of $C$ then the slice category $C_{/S}$ is again of the same type (with $\mathbf{P}$ replaced with the evident analogue for $C_{/S}$).

The following lemma illustrates the flexibility of hypercovers: we can "refine" in high degrees without changing low degrees. This provides for enormous flexibility in inductive proofs.

**Theorem 4.13.** *Fix an integer $n \geq 0$ and $C, \mathbf{P}$ as in one of the options above. For every simplicial object $X$ in $C$ for which $\mathrm{sk}_n X$ is split, there exists a map $f : X' \to X$ from another simplicial object $X'$ in $C$ with $\mathrm{sk}_n(f)$ an isomorphism and $X'$ split. In particular, every simplicial object in $C$ admits a "split refinement". All the same is true in the augmented case.*

*Moreover, if $X$ is an augmented $\mathbf{P}$-hypercover, we can take $X'$ to be an augmented $\mathbf{P}$-hypercover.*

It seems probable that the final part is false if one does not impose an augmentation structure.

*Proof.* The existence of a split refinement in the augmented case reduces to the non-augmented case by passing to a slice category, so now we only consider the nonaugmented case (until treating the last part of the theorem).

By recursion, to find $f : X' \to X$ with $\mathrm{sk}_n(f)$ an isomorphism and $X'$ split, it suffices to consider just $X$ which are $(n+1)$-truncated with $\mathrm{sk}_n^{n+1} X$ split, and to find $X'$ which is $(n+1)$-truncated and split and endowed with a map $X' \to X$ inducing an isomorphism on $n$-skeleta. Under the correspondence in Theorem 4.12, we simply take $X'$ to correspond to the triple $(\mathrm{sk}_n^{n+1} X, (\mathrm{cosk}_n^{n+1} \mathrm{sk}_n^{n+1} X)_{n+1}, \mathrm{id})$. Intuitively, $X'_{n+1}$ is a coproduct of $(\mathrm{cosk}_n^{n+1} \mathrm{sk}_n^{n+1} X)_{n+1}$ and various (repeated copies of the) $NX_j$'s for $j \leq n$. This takes care of the existence aspect of the theorem (and notice the construction is completely algorithmic).

To show the preservation of the $\mathbf{P}$-hypercover property when we're given augmentations, we can again reduce to the truncated situation and it suffices to show that in the construction just described, if the $(n+1)$-truncated $X$ (with $\mathrm{sk}_n X$ split) is a $\mathbf{P}$-hypercovering, then the $(n+1)$-truncated $X'$ is also a $\mathbf{P}$-hypercovering. The only issue is to study the degree $n+1$ map

$$\gamma : X'_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n^{n+1} X)_{n+1}.$$

We must show this map is of class $\mathbf{P}$. But by the very mechanism of construction of $X'_{n+1}$ above, $\gamma$ restricts to the identity on the component $NX'_{n+1} = (\mathrm{cosk}_n \mathrm{sk}_n^{n+1} X)_{n+1}$. This takes care of the topos case as well as the "surjective" aspects of the other cases. To handle the case of proper or étale hypercoverings, we need to see what $\gamma$ looks like when restricted to each component $NX_\phi$ with surjective $\phi : [n+1] \twoheadrightarrow [j]$ $(j \leq n)$. This restriction of $\gamma$ factors as

$$NX_j \to X_j = (\mathrm{cosk}_n \mathrm{sk}_n^{n+1} X)_j \xrightarrow{\phi} (\mathrm{cosk}_n \mathrm{sk}_n^{n+1} X)_{n+1}$$

(to see this, one has to actually look at the details of the construction in the proof of Theorem 4.12).

Since the surjective $\gamma$ factors as a composite of degeneracies, we just need to show that the degeneracies for $\mathrm{cosk}_n \mathrm{sk}_n^{n+1} X$ are of type $\mathbf{P}^0$, where $\mathbf{P}^0$ is defined much like $\mathbf{P}$ in each of our options except that we drop the surjectivity requirement. Since $\mathrm{cosk}_n \mathrm{sk}_n^{n+1} X$ is an augmented $\mathbf{P}$-hypercovering (see Example 4.5), it suffices to prove Corollary 4.14 below.

■

**Corollary 4.14.** *For $m \geq 0$, an* augmented *$m$-truncated $\mathbf{P}$-hypercovering $Z$ in the category of spaces, with $\mathbf{P}$ the class of either étale surjective or proper surjective maps, the face and degeneracy maps for $Z$ are automatically of type $\mathbf{P}^0$. The same holds in the non-truncated case.*

*Proof.* The non-truncated case is immediately reduced to the truncated case, so we assumed we're in the $m$-truncated case for some $m \geq 0$. Since each degeneracy map is a section to a face map, it suffices to show that the face maps are of type $\mathbf{P}^0$. The case $m = 0$ is clear, and to induct we may assume $m > 0$ and that all face and degeneracy maps for the split $Z' = \mathrm{sk}^m_{m-1} Z$ are of type $\mathbf{P}^0$. It remains to show that the face maps $Z_m \to Z_{m-1}$ are of type $\mathbf{P}^0$. By hypothesis on $Z$ being a $\mathbf{P}$-hypercovering,

$$Z_m \to (\mathrm{cosk}_{m-1} Z')_m$$

is of type $\mathbf{P}$ (hence $\mathbf{P}^0$). Thus, by naturality of adjunction it suffices to check that the face maps in $\mathrm{cosk}_{m-1} Z'$ from degree $m$ to degree $m-1$ are of type $\mathbf{P}^0$. Since $(\mathrm{cosk}_{m-1} Z')_m$ is constructed as an inverse limit on a finite diagram among the $Z'_j$'s, among which all maps are of type $\mathbf{P}^0$ (by the inductive hypothesis), the augmentation structure reduces us to proving that a finite diagram $D$ of spaces with type $\mathbf{P}^0$ transition maps has inverse limit with type $\mathbf{P}^0$ projection to each object in $D$ *provided* there is a final object $d_0$ in $D$ (this is completely false without a final object).

We argue by induction on the "size" of $D$, using only the stability properties of $\mathbf{P}^0$ and the fact that a section to a map of type $\mathbf{P}^0$ is again of type $\mathbf{P}^0$. If there are no arrows other than the ones to $d_0$, then the inverse limit is just the fiber product over $d_0$ and we're done. Otherwise there is an arrow $f : d_1 \to d_2$ with $d_2 \neq d_0$. If we remove this arrow (but not the objects $d_1$ and $d_2$), we get a "smaller" diagram $D'$ which satisfies all of the initial hypotheses, so by induction the inverse limit $L'$ of $D'$ has type $\mathbf{P}^0$ maps to all of its objects. In particular, the two maps $L' \to d_1$ and $L' \to d_2$ are type $\mathbf{P}^0$. The inverse limit over $D$ is exactly the inverse limit of the (not necessarily commutative) diagram

$$\begin{array}{ccc} L' & \longrightarrow & d_1 \\ & \searrow & \downarrow{\scriptstyle f} \\ & & d_2 \end{array}$$

in which all maps are of type $\mathbf{P}^0$ (the two from $L'$ to the $d_j$'s being of type $\mathbf{P}^0$ by the inductive hypothesis). We just have to show that the inverse limit of this diagram has type $\mathbf{P}^0$ projection to all three objects. In fact, we just have to check that the map to $L'$ is of type $\mathbf{P}^0$. The inverse limit is the fiber product

$$L = (d_1 \times_{d_2} L') \times_{d_1 \times_{d_2} d_1} d_1$$

with evident projections to $L'$, $d_1$, and $d_2$. Consider the composite

$$L \to d_1 \times_{d_2} L' \to L'.$$

The second is a base change on $f$ and hence is of type $\mathbf{P}^0$, while the first is a base change on the diagonal

$$d_1 \to d_1 \times_{d_2} d_1$$

which is itself of type $\mathbf{P}^0$ because it is a section to the (say, first) projection $d_1 \times_{d_2} d_1 \to d_1$ (which is a base change on $f$ and hence of type $\mathbf{P}^0$). ■

**Corollary 4.15.** *If $X_\bullet \to S$ is a proper (resp. étale) hypercovering, then all structure maps $X_n \to S$ are proper (resp. étale).*

It is now easy to prove Theorem 4.7, whose statement we recall.

**Theorem 4.16.** *Let $S$ be a separated scheme of finite type over a field $k$. Then there exists a dense open immersion $S \hookrightarrow \overline{S}$ into a proper $k$-scheme and an augmented proper hypercovering $\overline{X}_\bullet$ of $\overline{S}$ such that each $\overline{X}_n$ is a projective $k$-scheme which is regular (and hence is $k$-smooth for perfect $k$) and the part of $\overline{X}_n$ lying over $\overline{S} - S$ is a strict normal crossings divisor in $\overline{X}_n$ for all $n \geq 0$.*

*Proof.* By Nagata's compactification theorem [C], [L] (or just assume $S$ quasi-projective over $k$ if one wants to make restrictions), we can find a dense open immersion $S \hookrightarrow \overline{S}$ with $\overline{S}$ proper over $k$. By resolution of singularities (say in the form given by [dJ, 4.1]) applied to the (irreducible components of the) normalization of $\overline{S}$ and the complement of the preimage of the open $S$, we can find a regular $\overline{X}_0$ with a proper (even generically finite) surjection to $\overline{S}$ such that the preimage $X_0$ of $S$ in $\overline{X}_0$ has complement $D_0$ a strict normal crossings divisor. The solves the problem at the 0-truncated level.

Suppose we have solved the problem at the $m$-truncated level for some $m \geq 0$ with an augmented $X_{\leq m}$. By Example 4.5, $\operatorname{cosk}_m X_{\leq m}$ is a proper hypercovering of $S$. Thus, by Corollary 4.14, each term (e.g., the term in degree $m+1$) is $\overline{S}$-proper. Applying resolution of singularities again, now to the (normalization of the) term in degree $m+1$ gives a regular $\overline{X}'$ proper and generically finite over the $(m+1)$-coskeleton, with normal crossings divisor complement to the preimage of $S$. Now apply the construction in Theorem 4.13 to get an $(m+1)$-truncated solution. Continue forever. ■

## 5. Simplicial homotopy

Considering the historical origins of simplicial methods, it is hardly surprising that there should be a concept of homotopy for maps between simplicial objects, and that this should play a valuable role in the theory. One application of the homotopy concept in categories of simplicial objects is that it enables one to formulate Verdier's theorem on the computability of cohomology in a Grothendieck site by means of a direct limit over hypercovers, thereby "correcting" the failure of the direct limit (over ordinary open covers) of Čech cohomology to compute the true cohomology of a sheaf in degrees $> 1$ (and in fact, the hypercover result nicely clarifies why Čech theory does work in degrees $\leq 1$). The relevance of homotopies in this case is that (much like in Čech theory) it is only by passing to a suitable "homotopy category" of hypercovers that one can meaningfully pass to a direct limit over hypercovers. For our purposes, the significance of simplicial homotopy is that it is an essential ingredient in the proof of some of Deligne's theorems on cohomological descent.

We haven't yet discussed cohomological descent, or even the meaning of cohomology on simplicial spaces (or even what a "sheaf" on a simplicial space is!), so the applications of simplicial homotopy theory will have to wait until subsequent sections in which we give the necessary additional sheaf-theoretic concepts. Right now we want to just set forth the basic definition of homotopy and see how it interacts nicely with the coskeleton functor. The reader who wants to see what cohomological descent is and how it gives rise to spectral sequences can actually skip ahead to §6. It is only in §7 where we need to use homotopies: but we need this in order to actually prove some of the basic theorems on cohomological descent (e.g., that there exist lots of non-trivial examples!).

Consider the following basic setup. Let $X_\bullet$ and $X'_\bullet$ be (non-augmented) simplicial objects in an arbitrary category $C$, and let

$$f, g : X'_\bullet \to X_\bullet$$

be two maps between them. We want to define what it means to say that $f$ and $g$ are homotopic. To do this, for $m \geq 0$ recall the simplicial object $\Delta[m]$ in the category of (finite non-empty) sets, with $\Delta[m]_n = \operatorname{Hom}_\Delta([n],[m])$ and the evident contravariant functoriality in $[n]$. Corresponding to the two injective maps $[0] \to [1]$ we get two natural maps

$$\iota_0 : \Delta[0] \to \Delta[1], \ \ \iota_1 : \Delta[0] \to \Delta[1],$$

with $(\iota_j)_n$ sending the unique element in $\operatorname{Hom}_\Delta([n],[0])$ to the constant map $[n] \to [1]$ onto $j \in [1]$ for each degree $n \geq 0$. We view $\Delta[m]$ as the abstraction of the standard $m$-simplex, so the maps $\iota_j$ are viewed as the abstraction of the two identifications of a point with an endpoint of the unit interval.

From this point of view, we are motivated to want to define $f$ and $g$ to be homotopic if there exists a map

$$h : \Delta[1] \times X'_\bullet \to X_\bullet$$

such that composing $h$ with

$$X'_\bullet \simeq \Delta[0] \times X'_\bullet \stackrel{\iota_0 \times \mathrm{id}}{\longrightarrow} \Delta[1] \times X'_\bullet$$

yields $f$ and likewise using $\iota_1$ yields $g$. Of course, to make sense of this "definition" we need to make sense of $A_\bullet \times X'_\bullet$ as a simplicial object in $C$ whenever $A_\bullet$ is a simplicial object in the category of finite non-empty sets (e.g., $A_\bullet = \Delta[m]$ with $m \geq 0$). This is simple enough *if* we also assume $C$ has finite coproducts.

**Definition 5.1.** Assume $C$ admits finite coproducts. If $Y_\bullet$ is a simplicial object in $C$ and $A_\bullet$ is a simplicial object in the category of finite non-empty sets, we define the simplicial object $A_\bullet \times X_\bullet$ (also written $A \times X$) in $C$ as follows:

$$(A \times X)_n = A_n \times X_n := \coprod_{a \in A_n} X_n$$

as an object of $C$ (with the $a$-component denoted $\{a\} \times X_n$), and for $\phi : [n] \to [m]$ in $\Delta$ we define

$$\phi : (A \times X)_m \to (A \times X)_n$$

on "points" by

$$(a, x_m) \mapsto (A(\phi)(a), X(\phi)(x_m)).$$

It is trivial to check that this defines a simplicial object, is "associative" with respect to products in $A_\bullet$ and $X_\bullet$, and has evident bifunctoriality structure.

*Example* 5.2. For any $X_\bullet$, we naturally have $\Delta[0] \times X_\bullet = X_\bullet$, while $\iota_j : \Delta[0] \to \Delta[1]$ induces two natural maps

$$\iota_j \times \mathrm{id} : X_\bullet \simeq \Delta[0] \times X_\bullet \to \Delta[1] \times X_\bullet.$$

*Example* 5.3. Suppose $C$ is an abelian category, so finite coproducts are just finite direct sums. In this case, if $A$ is a simplicial object in $C$, then we describe $\Delta[1] \times A$ as follows.

In concrete terms, the elements of $\Delta[1]_n$ are naturally labelled by $0 \leq j \leq n+1$: one simply counts how often a non-decreasing map $[n] \to [1]$ hits 0. In this notation, an element in $(\Delta[1] \times A)_n$ is a tuple $(a_0, \dots, a_{n+1})$ of elements of $A_n$ and the inclusions $\iota_0, \iota_1$ correspond in degree $n$ to the embeddings

$$A_n \to \prod_{j=0}^{n+1} A_n$$

given by $a \mapsto (0, \dots, 0, a)$ and $a \mapsto (a, 0, \dots, 0)$ respectively.

For $0 \leq i \leq n$, the $i$th degree $n$ face map on $\Delta[1] \times A$ is

$$\partial_n^i(a_0, \dots, a_{n+1}) = (\partial_n^i(a_0), \dots, \partial_n^i(a_i) + \partial_n^i(a_{i+1}), \dots, \partial_n^i(a_{n+1}))$$

and the $i$th degree $n$ degeneracy map on $\Delta[1] \times A$ is

$$\sigma_n^i(a_0, \dots, a_{n+1}) = (\sigma_n^i(a_0), \dots, \sigma_n^i(a_i), 0, \sigma_n^i(a_{i+1}), \dots, \sigma_n^i(a_{n+1})).$$

*Example* 5.4. If $F : C \to C'$ is a covariant functor which converts finite coproducts into finite coproducts, then there is a natural isomorphism

$$F(A \times X_\bullet) \simeq A \times F(X_\bullet)$$

for any simplicial object $A$ in the category of finite non-empty sets.

With the product language in Definition 5.1, we can define homotopy:

**Definition 5.5.** Assume $C$ admits finite coproducts. For two maps $f, g : X'_\bullet \to X_\bullet$ between simplicial objects of $C$, we say that a map $h : \Delta[1] \times X'_\bullet \to X_\bullet$ satisfying $h \circ \iota_0 = f$ and $h \circ \iota_1 = g$ is a *strict homotopy from $f$ to $g$*. If there exists a strict homotopy from $f$ to $g$, or from $g$ to $f$, we say $f$ and $g$ are *strictly homotopic*.

If there exist $r \geq 1$ and maps

$$f = F_0, F_1, \dots, F_r = g : X'_\bullet \to X_\bullet$$

such that for each $0 \leq j < r$, the maps $F_j$ and $F_{j+1}$ are strictly homotopic, we say that $f$ and $g$ are *homotopic*.

Clearly homotopy is an equivalence relation, and by Example 5.4 we see that any covariant functor $F : C \to C'$ commuting with finite coproducts carries homotopic maps to homotopic maps. It is actually possible to reformulate the definition of simplicial homotopy in a way that makes sense in an arbitrary category (i.e., without requiring the existence of finite coproducts) and for which covariant functors always preserve the property of being homotopic. To define a strict homotopy from $f$ to $g$ more generally for two maps $f, g$ as above (i.e., without using coproducts), one specifies a sequence of set maps

$$H_n : \Delta[1]_n \to \mathrm{Hom}_C(X'_n, X_n)$$

which carry the contravariant functoriality in $[n]$ on the left over to the the co/contra-variant bifunctoriality in $[n]$ on the right, and moreover satisfy

$$H_n((\iota_0)_n) = f_n \quad H_n((\iota_1)_n) = g_n,$$

where $(\iota_j)_n \in \Delta[1]_n = \mathrm{Hom}_\Delta([n], [1])$ is the constant map onto $j \in [1]$.

The functoriality conditions amount to saying that for each $\phi : [i] \to [i']$ in $\Delta$ we require the diagram

$$\begin{array}{ccccc} \Delta[1]_{i'} & \xrightarrow{H_{i'}} & \mathrm{Hom}_C(X'_{i'}, X_{i'}) & \xrightarrow{X(\phi)} & \mathrm{Hom}_C(X'_{i'}, X_i) \\ {\scriptstyle \Delta[1](\phi)}\downarrow & & & \nearrow {\scriptstyle X'(\phi)} & \\ \Delta[1]_i & \xrightarrow[H_i]{} & \mathrm{Hom}_C(X'_i, X_i) & & \end{array} \tag{5.1}$$

to commute. It is a simple exercise to check that this recovers the previous definition of strict homotopy from $f$ to $g$ when $C$ admits finite coproducts. It is also clear from (5.1) that any covariant functor $F : C \to C'$ carries homotopic maps to homotopic maps (and likewise for contravariant $F$ if we use cosimplicial objects in $C'$).

There are many reasons for interest in this concept of simplicial homotopy, but we only mention the ones we'll require.

**Lemma 5.6.** *Let $f, g : X'_\bullet \to X_\bullet$ be two homotopic maps between simplicial objects in a category $C$. If $F : C \to \mathscr{A}$ is any covariant functor to an abelian category, then the cochain complex maps*

$$\mathbf{s}(F(f)), \mathbf{s}(F(g)) : \mathbf{s}(F(X'_\bullet)) \to \mathbf{s}(F(X_\bullet))$$

*are homotopic in the usual sense, where $\mathbf{s}$ is the functor which makes a cochain complex by using the alternating sums of face maps as differentials.*

*A similar statement holds for contravariant functors.*

*Proof.* The contravariant case is reduced to the covariant case by replacing $\mathscr{A}$ with the opposite category, so we may just consider the covariant case.

Since the property of being homotopic is preserved under applying a covariant functor, we reduce to the case of in which $C = \mathscr{A}$ is an abelian category and $F$ is the identity functor. Since homotopy on the level of cochain complexes is (unlike strict simplicial homotopy) an equivalence relation, it suffices to show that for any simplicial object $A$ in $\mathscr{A}$, the two inclusions

$$A \simeq \Delta[0] \times A \to \Delta[1] \times A$$

become homotopic in the usual sense after we apply $\mathbf{s}$. We will use the description of $\Delta[1] \times A$ provided by Example 5.3.

The degree $n$ differential on the cochain complex $\mathbf{s}(\Delta[1] \times A)$ is

$$\partial(a_0, \dots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i (\partial_n^i(a_0), \dots, \partial_n^i(a_i) + \partial_n^i(a_{i+1}), \dots, \partial_n^i(a_{n+1})).$$

We want to find a (functorial) homotopy between the two maps

$$\mathbf{s}(\iota_j) : \mathbf{s}(A) \simeq \mathbf{s}(\Delta[0] \times A) \to \mathbf{s}(\Delta[1] \times A)$$

defined by

$$a \mapsto (a, 0, \dots, 0), \ \ a \mapsto (0, \dots, 0, a)$$

in each degree. Such a homotopy is provided by the maps

$$h_n : A_n \to (\Delta[1] \times A)_{n+1} = \prod_{j=0}^{n+2} A_{n+1}$$

given by

$$a \mapsto (0, s_0(a), -s_1(a), \dots, (-1)^i s_i(a), \dots, (-1)^n s_n(a), 0),$$

and in fact one checks with a bit of computation that

$$h\partial + \partial h = \iota_1 - \iota_0$$

(where $\iota_1$ is inclusion into the 0th coordinate and $\iota_0$ is inclusion into the final coordinate). ■

The other fact we'll need concerning simplicial homotopies is their behavior on coskeleta (so we assume $C$ also admits finite products and fiber products).

**Lemma 5.7.** *If two maps $f, g : X'_\bullet/S \to X_\bullet/S$ between augmented $n$-truncated objects in $C$ agree on $(n-1)$-skeleta for some $n \geq 0$ (a vacuous condition if $n = 0$), then there exists a canonical strict simplicial homotopy (over $S$) between $\mathrm{cosk}_n(f)$ and $\mathrm{cosk}_n(g)$. In particular, if*

$$f : X'_\bullet/S \to X_\bullet/S, \ \ s : X_\bullet/S \to X'_\bullet/S$$

*are maps of augmented $n$-truncated objects in $C$ for which $f$ and $s$ are inverses on $(n-1)$-skeleta and $s_n$ is a section to $f_n$, then $\mathrm{cosk}_n(f)$ and $\mathrm{cosk}_n(s)$ are strict simplicial homotopy inverses of each other (i.e., composites in either order are strictly homotopic to the identity map).*

*This all remains valid in the non-augmented case as well.*

*Proof.* Passing to a slice category, we can drop the augmentation data. For $0 \leq i \leq n$, we define

$$H_i : \Delta[1]_i = \mathrm{Hom}_\Delta([i], [1]) \to \mathrm{Hom}_C(X'_i, X_i)$$

by $H_i((\iota_0)_i) = f_i$ and $H_i(\phi) = g_i$ for all $\phi : [i] \to [1]$ not equal to the constant map $(\iota_0)_i$ onto $0 \in [1]$ (so for $0 \leq i < n$ we have $H_i$ is the constant map to $f_i = g_i$). A little thought using *functoriality* of $f$ and $g$ in degrees $\leq n$ shows that the resulting diagrams (5.1) do commute for $i, i' \leq n$.

For $m > n$, the construction of $\mathrm{cosk}_n$ yields

$$X'_m = \varprojlim_{\mathrm{sk}_n(\Delta[m])} X'_\phi, \ \ X_m = \varprojlim_{\mathrm{sk}_n(\Delta[m])} X_\phi$$

as inverse limits over all maps $\phi : [i] \to [m]$ with $i \leq n$. Thus, there is a natural map of sets

$$\varprojlim_{\mathrm{sk}_n(\Delta[m])} \mathrm{Hom}_C(X'_\phi, X_\phi) \to \mathrm{Hom}_{\mathrm{Simp}_n(C)}(X'_m, X_m).$$

For $m > n$, one easily checks

$$\Delta[1]_m = \mathrm{Hom}_\Delta([m], [1]) = \varprojlim_{\mathrm{sk}_n(\Delta[m])} \mathrm{Hom}_\Delta([\phi], [1])$$

where $[\phi] = [i]$ for $\phi : [i] \to [m]$ with $i \leq n$. For example, if $\phi_1, \phi_2 : [m] \to [1]$ are distinct then $m \geq 1$ and there exists a map $[0] \to [m]$ whose composites with the $\phi_j$'s are distinct. If we define $H_\phi = H_i$ for $\phi : [i] \to [m]$ with $i \leq n$, then we get a natural map of sets

$$H_m = \varprojlim_{\mathrm{sk}_n(\Delta[m])} H_\phi : \Delta[1]_m \to \varprojlim_{\mathrm{sk}_n(\Delta[m])} \mathrm{Hom}_C(X'_m, X_\phi) \ = \ \mathrm{Hom}_C(X'_m, X_m),$$

where the initial map of inverse limit sets is defined by means of passage to the limit on the maps

$$\Delta[1]_m \xrightarrow{\phi} \Delta[1]_i \xrightarrow{H_i} \mathrm{Hom}_C(X'_i, X_i) \xrightarrow{X'(\phi)} \mathrm{Hom}_C(X'_m, X_i)$$

for $\phi : [i] \to [m]$ with $i \leq n$.

One checks without difficulty that this inverse limit construction recovers the preceding definition of $H_m$ in case $m \leq n$ and that it is functorial in $[m]$. Also, it is easy to check that $H_m((\iota_0)_m) = f_m$ and $H_m((\iota_1)_m) = g_m$ for all $m$, so the $H_m$'s provided the desired strict homotopy.

The *naturality* of this construction is formulated as follows. Consider a commutative diagram

$$\begin{array}{ccc} X'_\bullet/S & \overset{f}{\underset{g}{\rightrightarrows}} & X_\bullet/S \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ Y'_\bullet/S & \overset{F}{\underset{G}{\rightrightarrows}} & Y_\bullet/S \end{array}$$

with $\mathrm{sk}_{n-1}(F) = \mathrm{sk}_{n-1}(G)$ too. Let $H(f,g)$ and $H(F,G)$ denote the homotopies arising from the preceding construction. The compatibility of these can be expressed by means of the commutative diagram

$$\begin{array}{ccc} \Delta[1] \times X'_\bullet & \xrightarrow{H(f,g)} & X \\ {\scriptstyle 1\times u'}\downarrow & & \downarrow{\scriptstyle u} \\ \Delta[1] \times Y'_\bullet & \xrightarrow[H(F,G)]{} & Y \end{array}$$

when $C$ has finite coproducts, and more generally by means of the commutativity of

$$\begin{array}{ccc} \Delta[1]_i & \xrightarrow{H_i(f,g)} & \mathrm{Hom}_C(X'_i, X_i) \\ {\scriptstyle H_i(F,G)}\downarrow & & \downarrow{\scriptstyle u_i} \\ \mathrm{Hom}_C(Y'_i, Y_i) & \xrightarrow[u'_i]{} & \mathrm{Hom}_C(X'_i, Y_i) \end{array}$$

for all $i$. The verification of these commutativities is straightfoward from the construction.

■

## 6. Cohomological descent

The reader who wishes to only think about spaces in our usual sense (i.e., the category of topological spaces, or schemes with the étale topology) can take that point of view without impacting any arguments. However, we note here that our arguments only require that we work with a Grothendieck site admitting finite fiber products, with the topology generated by a class $E$ of maps satisfying the habitual axioms (preserved under base change and composition, containing all isomorphisms, etc.). When we speak of "spaces" and "étale maps", the reader who prefers this extra generality should just interpret "space" to mean an object in the site and an "étale map" to mean an $E$-morphism.

Fix a simplicial object $X_\bullet$ in our category $C$ of spaces, or an $m$-truncated such object in $C$ for some $m \geq 0$. We would like to define what we mean by "sheaf of sets on $X_\bullet$". Intuitively, such structures should amount to specifying a sheaf of sets $\mathscr{F}^n$ on each $X_n$ (the category of which is denoted $\widetilde{X_n}$), along with transition maps as in (2.4), satisfying the obvious face/degeneracy relations. More exhaustively, for any $\phi : [n] \to [m]$ in $\Delta$ (so $X(\phi) : X_m \to X_n$), we specify a map of sheaves

$$[\phi] = [\phi]_{\mathscr{F}^\bullet} : X(\phi)^*(\mathscr{F}^n) \to \mathscr{F}^m$$

and we require

$$[\phi] \circ X(\phi)^*[\psi] = [\phi \circ \psi]$$

for all composable $\phi, \psi$ (so these $\mathscr{F}^\bullet$'s are vaguely "cosimplicial"). By the usual argument, it suffices to define $[\phi]$ for $\phi$ a face or degeneracy map, subject to the usual relations. These form a category in the evident manner.

For technical reasons, it is convenient to also make the following alternative (but equivalent) definition.

**Definition 6.1.** Define $\widetilde{X_\bullet}$ to be the category of sheaves of sets on the following site:

- the objects are étale maps $U \to X_n$
- a morphism from $(U \to X_n)$ to $(U' \to X_{n'})$ is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ \downarrow & & \downarrow \\ X_n & \xrightarrow[X(\phi)]{} & X_{n'} \end{array}$$

  where $f$ is a map in $C$ and $\phi : [n'] \to [n]$ is a map in $\Delta$,
- a covering of $(U_i \to X_n)$ is just a covering of $U_i$ in $C$.

We call such objects *sheaves of sets on* $X_\bullet$. We write $X_\bullet$ to denote this site, and $\mathrm{Ab}(X_\bullet)$ to denote the subcategory of abelian group objects in $\widetilde{X_\bullet}$.

*Example* 6.2. If $X_\bullet = S_\bullet$ is a constant (non-augmented) simplicial object on $S$, then $\widetilde{X_\bullet} = \mathrm{Cosimp}(\widetilde{S})$ and $\mathrm{Ab}(S_\bullet) = \mathrm{Cosimp}(\mathrm{Ab}(S))$.

The reader can readily check that this fancy-looking definition gives the exact same category as in the more explicit definition initially suggested. The reader who prefers to avoid thinking about this mild site can adopt the more naive definition first suggested. However, there are certain technical points where the naive definition becomes a bit of a headache. For example, from the point of view of the naive definition (i.e., without thinking in terms of the site) it is not obvious that the category $\mathrm{Ab}(X_\bullet)$ has enough injectives. The nuisance is that a given sheaf on some $X_n$ has no evident natural way to propograte itself to a sheaf on $X_\bullet$. But from the viewpoint of the site introduced above, this problem goes away because a general argument shows that the category of abelian sheaves on any site always has enough injectives. On the other hand, even once we know abstract existence it is hard to see any simple criterion for whether a given abelian $\mathscr{F}^\bullet$ is injective in $\mathrm{Ab}(X_\bullet)$ in terms of the $\mathscr{F}^n$'s and maps between their various pullbacks.

Observe that sheafification relative to this site can be done degree-by-degree, and likewise images, quotients by equivalence relations, and equalizer kernels of maps of sheaves on $X_\bullet$ can be computed degree-by-degree (as one readily checks the universal properties). Similarly, given a diagram in $\widetilde{X_\bullet}$ we can form its inverse limit and direct limit in $\widetilde{X_\bullet}$ by using the constructions in the individual $\widetilde{X_n}$'s. All this is just saying that the restriction functors $\widetilde{X_\bullet} \to \widetilde{X_n}$ on categories of sheaves of sets do commute with formation of images, equalizer kernels, and so on.

In order to pass between sheaves on $X_\bullet$ and sheaves on the $X_n$'s, we record the following two lemmas. The first lemma is an immediate consequence of working locally on the site $X_\bullet$ (which is sufficient for checking monicity and epicity of a morphism of sheaves of sets).

**Lemma 6.3.** *A map $\alpha^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ in $\widetilde{X_\bullet}$ is epic if and only if each $\alpha^n : \mathscr{F}^n \to \mathscr{G}^n$ in $\widetilde{X_n}$ is epic. The same holds for the properties of being monic and being an isomorphism.*

As we noted, it seems rather difficult to say what an injective object in $\mathrm{Ab}(X_\bullet)$ looks like. Fortunately, for our later purposes we only need to know the first thing that comes to mind on this topic. This innocent little fact will be extremely useful.

**Lemma 6.4.** *Let $\mathscr{I}^\bullet$ be an injective object in* $\mathrm{Ab}(X_\bullet)$. *Then $\mathscr{I}^n$ is an injective in* $\mathrm{Ab}(X_n)$ *for every* $n \geq 0$.

The intuition here comes from the case of Čech hypercoverings, for which this lemma is vaguely analogous to the fact that an injective abelian sheaf restricts to an injective on any "open" in the space. This classical fact is proven by means of a left exact "extension by zero" functor which is left adjoint to the restriction functor. We simply create an analogous adjoint functor and check left exactness in the simplicial context.

A bisimplicial version of this lemma, concerning restriction to a row or a column for a "sheaf on a bisimplicial object", is given in Lemma 7.6 and will be rather important later.

*Proof.* By general nonsense, it suffices to show that the restriction functor $\mathrm{Ab}(X_\bullet) \to \mathrm{Ab}(X_n)$ has a *left exact* left adjoint

$$L_n = L_n^{X_\bullet} : \mathrm{Ab}(X_n) \to \mathrm{Ab}(X_\bullet)$$

for each $n \geq 0$. That is, for any abelian $\mathscr{G}^\bullet$ on $X_\bullet$ and abelian $\mathscr{F}$ on $X_n$, we seek to find some $L_n(\mathscr{F})$ in $\mathrm{Ab}(X_\bullet)$ such that

$$\mathrm{Hom}_{\mathrm{Ab}(X_n)}(\mathscr{F}, \mathscr{G}^n) \simeq \mathrm{Hom}_{\mathrm{Ab}(X_\bullet)}(L_n(\mathscr{F}), \mathscr{G}^\bullet)$$

in a bifunctorial manner, with $\mathscr{F} \rightsquigarrow L_n(\mathscr{F})$ left exact.

For any $\mathscr{F}$ in $\mathrm{Ab}(X_n)$, we first want to show that the covariant functor

$$H_{\mathscr{F}} : \mathscr{G}^\bullet \rightsquigarrow \mathrm{Hom}_{\mathrm{Ab}(X_n)}(\mathscr{F}, \mathscr{G}^n)$$

is co-representable on $\mathrm{Ab}(X_n)$ (i.e., has the form $\mathrm{Hom}_{\mathrm{Ab}\, X_\bullet}(L_n(\mathscr{F}), \cdot)$ for some $L_n(\mathscr{F})$ in $\mathrm{Ab}(X_\bullet)$). This will provide the desired adjoint functor $L_n$, and we will have to make sure it preserves left exactness.

For each $m \geq 0$, we define the sheaf $L_n(\mathscr{F})^m$ as a finite direct sum:

$$L_n(\mathscr{F})^m = \bigoplus_{\phi:[n]\to[m]} X(\phi)^*(\mathscr{F}).$$

For any map $\psi : [m] \to [m']$ in $\Delta$, the composite $\psi \circ \phi : [n] \to [m']$ is one of the maps in the direct sum defining $L_n(\mathscr{F})^{m'}$, so by means of the canonical isomorphism $X(\psi)^* \circ X(\phi)^* \simeq X(\psi \circ \phi)^*$ we get a natural composite

$$[\psi] : X(\psi)^*(L_n(\mathscr{F})^m) = \bigoplus_{\phi:[n]\to[m]} X(\psi \circ \phi)^*(\mathscr{F}) \to \bigoplus_{\phi':[n]\to[m']} X(\phi')^*(\mathscr{F}) = L_n(\mathscr{F})^{m'}.$$

It is straightfoward to check that for any $\psi' : [m'] \to [m'']$ in $\Delta$ we have an equality of maps

$$[\psi'] \circ X(\psi')^*[\psi] = [\psi' \circ \psi],$$

so this data is exactly the original "naive" definition of a sheaf on $X_\bullet$. We denote this sheaf $L_n(\mathscr{F})^\bullet$. This construction has evident (additive) functoriality in $\mathscr{F}$ with respect to which it is trivially exact, as exactness can be checked in each separate degree.

Since $\mathscr{F}$ is naturally one of the "components" of $L_n(\mathscr{F})^n$ (corresponding to the identity map on $[n]$), composition with $\mathscr{F} \to L_n(\mathscr{F})^n$ sets up a bifunctorial map

(6.1) $$\mathrm{Hom}_{\mathrm{Ab}(X_\bullet)}(L_n(\mathscr{F})^\bullet, \mathscr{G}^\bullet) \to \mathrm{Hom}_{\mathrm{Ab}(X_n)}(L_n(\mathscr{F})^n, \mathscr{G}^n) \to \mathrm{Hom}_{\mathrm{Ab}(X_n)}(\mathscr{F}, \mathscr{G}^n)$$

which we will show to be bijective, thereby completing the proof that the (exact) functor $L_n$ is the desired left adjoint.

To show (6.1) is bijective, we construct an inverse. Pick a map of abelian sheaves $\xi : \mathscr{F} \to \mathscr{G}^n$ on $X_n$. For any $\phi : [n] \to [m]$ in $\Delta$ we get composite maps

$$X(\phi)^*(\mathscr{F}) \xrightarrow{X(\phi)^*(\xi)} X(\phi)^*(\mathscr{G}^n) \xrightarrow{[\phi]_{\mathscr{G}^\bullet}} \mathscr{G}^m ,$$

with the last map coming from the description of the abelian sheaf $\mathscr{G}^\bullet$ on $X_\bullet$ in "naive" terms. Taking these maps over all $\phi : [n] \to [m]$ with fixed $m$ then defines a map of sheaves

$$\xi^m : L_n(\mathscr{F})^m \to \mathscr{G}^m.$$

Unwinding the compatbility properties of the structure maps $[\psi]_{\mathscr{G}^\bullet}$ of $\mathscr{G}^\bullet$ for all $\psi$ in $\Delta$ readily yields that for any $\psi : [m] \to [m']$ in $\Delta$, the diagram

$$\begin{array}{ccc} X(\psi)^*(L_n(\mathscr{F})^m) & \xrightarrow{X(\psi)^*(\xi^m)} & X(\psi)^*(\mathscr{G}^m) \\ \downarrow & & \downarrow \\ L_n(\mathscr{F})^{m'} & \longrightarrow & \mathscr{G}^{m'} \end{array}$$

commutes. Hence, we have constructed a map of abelian sheaves on $X_\bullet$ from $L_n(\mathscr{F})^\bullet$ to $\mathscr{G}^\bullet$.

It is clear that this construction

$$\mathrm{Hom}_{\mathrm{Ab}(X_n)}(\mathscr{F},\mathscr{G}^n) \to \mathrm{Hom}_{\mathrm{Ab}(X_\bullet)}(L_n(\mathscr{F})^\bullet,\mathscr{G}^\bullet) \tag{6.2}$$

is bifunctorial, and that (6.2) followed by (6.1) is the identity on $\mathrm{Hom}_{\mathrm{Ab}(X_n)}(\mathscr{F},\mathscr{G}^n)$. To check the composite in the other direction is the identity, we have to show that an arbitrary map of sheaves $L_n(\mathscr{F})^\bullet \to \mathscr{G}^\bullet$ is uniquely determined by the induced map $\mathscr{F} \to \mathscr{G}^n$ within degree $n$, and more specifically is obtained from this induced map within degree $n$ by means of the construction which produced (6.2). This is straightfoward definition-chasing, using the compatibilities for the structure maps $[\psi]_{\mathscr{G}^\bullet}$.

■

For a map $u_\bullet : X_\bullet \to Y_\bullet$ of simplicial spaces (without augmentation!), or $m$-truncated simplicial spaces with some $m \geq 0$, we can use the usual term-by-term constructions of pushfoward and pullback to define functors

$$u_{\bullet *} : \widetilde{X_\bullet} \to \widetilde{Y_\bullet},\ \ u_\bullet^* : \widetilde{Y_\bullet} \to \widetilde{X_\bullet}$$

with the usual adjointness and exactness properties (recall that exactness aspects can be checked in each separate degree). Thus, in fancy terms, this defines a morphism of topoi. For our purposes, the case of interest will be an augmented simplicial (or $m$-truncated simplicial, with $m \geq 0$) space $a : X_\bullet \to S$. In this case, if we let $S_\bullet$ denote the constant simplicial space attached to $S$, there is a unique map

$$a_\bullet : X_\bullet \to S_\bullet$$

respecting the augmentations. Observe that $\widetilde{S_\bullet}$ is canonically identified with the category $\mathrm{Cosimp}(\widetilde{S})$ of cosimplicial sheaves of sets on $S$. Even though $a$ isn't really a "map", we can still define adjoint functors which we'll call $a_*$ and $a^*$ between $\widetilde{X_\bullet}$ and $\widetilde{S}$ (and so from the point of view of topoi, we really do have a "map" $a$ after all).

There is a natural (exact) pullback functor

$$a^* : \widetilde{S} \to \widetilde{X_\bullet}$$

defined by $(a^*\mathscr{F})^n = a_n^*\mathscr{F}$ with the evident "face" and "degeneracy" maps just as in Example 2.8. This visibly left exact functor is an honest pullback in that it does have a right adjoint

$$a_* : \widetilde{X_\bullet} \to \widetilde{S}$$

given by defining $a_*\mathscr{F}^\bullet$ to be the kernel equalizer of

$$a_{0*}\mathscr{F}^0 \underset{\sigma_1^1}{\overset{\sigma_0^1}{\rightrightarrows}} a_{1*}\mathscr{F}^1$$

(and since $a^*$ has a right adjoint, it is not only left exact but also right exact; if $X_\bullet$ is 0-truncated then $a_*\mathscr{F}^\bullet$ is just pushfoward by $X_0 \to S$). To check this adjointness, we could compute directly or we can argue more elegantly as follows. If we do the same construction $(\varepsilon_S^*, \varepsilon_{S*})$ for the constant augmented simplicial space $\varepsilon_S : S_\bullet \to S$, it is trivial to check that

$$a_* \simeq \varepsilon_{S*} \circ a_{\bullet *},\ \ a^* \simeq a_\bullet^* \circ \varepsilon_S^*.$$

Hence, to check adjointness we are reduced to the case $X_\bullet = S_\bullet$ with its constant augmentation, for which $\widetilde{S_\bullet} = \mathrm{Cosimp}(\widetilde{S})$ and hence everything is clear "by hand" (and of course one gets the adjunction maps one expects).

On the level of abelian sheaves, we get derived functors

$$a^* : \mathbf{D}_+(S) \to \mathbf{D}_+(X_\bullet),\ \ \mathbf{R}a_* : \mathbf{D}_+(X_\bullet) \to \mathbf{D}_+(S)$$

by the usual arguments. Now we can give the key definition of these notes.

**Definition 6.5.** The adjoint pair $(a_*, a^*) : \widetilde{X_\bullet} \to \widetilde{S}$ (which we'll often abbreviate by writing $a : X_\bullet \to S$) is said to be a *morphism of cohomological descent* if the natural transformation

$$\mathrm{id} \to \mathbf{R}a_* \circ a^*$$

on $\mathbf{D}_+(S)$ is an isomorphism.

This concept is called a 1-descent morphism in [SGA4]. It corresponds to just the full faithfulness aspect of classical descent theory (as we'll see in a moment), whereas incorporating a further derived analogue of effectivity gives rise to the stronger notion called a 2-descent morphism in [SGA4] about which we'll say nothing here.

*Remark* 6.6. The reader will observe that whenever we have worked with abelian sheaves, we could just as well have worked with sheaves of $R$-modules for a fixed commutative ring $R$ (and likewise for the derived categories). We opted to stick with the case $R = \mathbf{Z}$ for expository simplicity. There is really only one place where this makes a difference, namely in the case of proper hypercoverings for the étale site on schemes. In that theory, one only has the proper base change for higher direct image sheaves when $R$ is a torsion ring, such as $R = \mathbf{Z}/n$ for some positive integer $n$. In [SGA4] the coefficient sheaf of rings is axiomatized at the beginning so as to treat all cases at the same time. We'll continue to work with $R = \mathbf{Z}$, except in a couple of places where we state results for proper hypercoverings on the étale site, in which case we'll pick $R = \mathbf{Z}/n$ (but any torsion ring would do just fine). The reader familiar with the étale site will readily see how it all extends to torsion sheaves and $\ell$-adic sheaves (e.g., functoriality is checked for spectral sequences in the torsion sheaf case, and then one can pass to limits to get spectral sequences in the $\ell$-adic case), so we omit discussion in that direction.

Note that by standard exact triangle arguments, the condition of being of cohomological descent on the level of derived categories is equivalent to the following assertion on the level of abelian sheaves: for any $\mathscr{F}$ in $\mathrm{Ab}(S)$, we should have

$$\mathscr{F} \simeq a_* a^* \mathscr{F} = \ker(a_{0*} a_0^* \mathscr{F} \to a_{1*} a_1^* \mathscr{F}),\ \ \mathrm{R}^i\, a_*(a^* \mathscr{F}) = 0 \text{ for all } i > 0$$

(where the kernel involves the difference of the two "cosimplicial" face maps). It is the vanishing of the $\mathrm{R}^i\, a_*$'s which "distinguishes" cohomological descent from ordinary descent theory, though such vanishing seems to have no down-to-earth meaning. Instead, the true "meaning" of cohomological descent is perhaps best captured by Lemma 6.8 (and the discussion preceding it). We'll make the link with descent theory a bit more explicit following the next example.

*Example* 6.7. The most elementary example of cohomological descent is the case of the augmentation $S_\bullet \to S$ from the constant simplicial space on $S$. In this case, $a_*$ is essentially the $\mathrm{H}^0$ functor under the identification $\mathrm{Ab}(S_\bullet) = \mathrm{Cosimp}(\mathrm{Ab}(S))$. Now under the $\mathrm{H}^0$-compatible (!) Dold-Kan correspondence

$$\mathrm{Cosimp}(\mathrm{Ab}(S)) \simeq \mathrm{Ch}_{\geq 0}(\mathrm{Ab}(S)),$$

the adjoint to

$$a_* = \mathrm{H}^0 : \mathrm{Ch}_{\geq 0}(\mathrm{Ab}(S)) \to \mathrm{Ab}(S)$$

is just the functor $\mathscr{F} \mapsto \mathscr{F}[0]$. The adjunction $\mathrm{id} \to a_* a^*$ is readily checked to be the canonical map

$$\mathscr{F} \mapsto \mathrm{H}^0(\mathscr{F}[0]),$$

and this is an isomorphism. Since the derived functors of $\mathrm{H}^0$ are exactly $\mathrm{H}^j$, we see that

$$\mathrm{R}^j\, a_*(a^* \mathscr{F}) = \mathrm{H}^j(\mathscr{F}[0]) = 0$$

for $j > 0$. Since the adjunction $\mathscr{F} \to a_* a^* \mathscr{F}$ is an isomorphism, we conclude that indeed $a$ is of cohomological descent in the constant case.

Let's now see the reason for the terminology "cohomological descent". In descent theory, one has a "cover" $p : X' \to X$ and introduces the two projections

$$p_0, p_1 : X'' := X' \times_X X' \to X'.$$

Given a sheaf $\mathscr{F}$ on $X$, we then get the pair of data

$$(\mathscr{F}' = p^* \mathscr{F}, \alpha : p_1^* \mathscr{F}' \simeq p_2^* \mathscr{F}')$$

consisting of a sheaf on $X'$ and an isomorphism between its two natural pullbacks to $X''$ *such that* $\alpha$ enjoys an additional cocycle compatibility

$$p_{02}^*(\alpha) = p_{12}^*(\alpha) \circ p_{01}^*(\alpha)$$

when we pull it back to the triple fiber power. This sort of structure is very closely related to the data of a sheaf on $\mathrm{sk}_2 \,\mathrm{cosk}_0(X'/X)$ (to see this, look at the calculations in [BLR, p. 133]). Classical descent theory says (under certain conditions on $p$, depending on the geometric category in which one is working) that the functor $\mathscr{F} \rightsquigarrow (\mathscr{F}', \alpha)$ is *fully faithful*. Note that a map

$$(\mathscr{F}', \alpha) \to (\mathscr{G}', \beta)$$

between two such pairs of data is really just a map on the level of sheaves on the 1-skeleton $\mathrm{sk}_1 \,\mathrm{cosk}_0(X'/X)$, hence the terminology "1-descent morphism" in [SGA4].

The cohomological descent property is essentially a "derived" version of *full faithfulness*, where we work on an entire simplicial object and not just its truncations in degrees $\leq 2$:

**Lemma 6.8.** *A map $a : X_\bullet \to S$ is a morphism of cohomological descent if and only if $a^* : \mathbf{D}_+(S) \to \mathbf{D}_+(X_\bullet)$ is fully faithful.*

*Proof.* To say $\mathrm{id} \to \mathbf{R}a_* \circ a^*$ is an isomorphism is to say that the adjunction

$$K \to \mathbf{R}a_*(a^*K)$$

is an isomorphism for every complex $K$ in $\mathbf{D}_+(S)$. Equivalently, by Yoneda, this says that for each $K'$ in $\mathbf{D}_+(S)$, the natural map

$$\mathrm{Hom}(K', K) \to \mathrm{Hom}(K', \mathbf{R}a_*(a^*K))$$

is bijective. But due to the adjointness between $\mathbf{R}a_*$ and $a^*$ on (bounded below) derived categories, we get a commutative triangle

$$\begin{array}{ccc} \mathrm{Hom}(K', K) & \longrightarrow & \mathrm{Hom}(K', \mathbf{R}a_* a^* K) \\ & \searrow^{a^*} & \downarrow \simeq \\ & & \mathrm{Hom}(a^*K', a^*K) \end{array}$$

in which the diagonal arrow is the functor $a^*$. The desired equivalence follows. ∎

*Example* 6.9. We'll see in §7 that both proper hypercoverings and étale hypercoverings (really hypercoverings for any Grothendieck topology) are of cohomological descent. We stress that even the special case of a 0-coskeleton on a proper or étale surjective map $X_0 \to S$ is not obvious.

In order to formulate the spectral sequence relating cohomology on $S$ with that on the $X_p$'s in the presence of a morphism of cohomological descent $a : X_\bullet \to S$, we first need to define a "global sections" functor on the simplicial object $X_\bullet$. This will have nothing to do with the augmentation structure, and goes as follows.

**Definition 6.10.** Let $X_\bullet$ be a simplicial space (without augmentation!). For an abelian sheaf $\mathscr{F}^\bullet$ on $X_\bullet$, we define the *global sections* of $\mathscr{F}^\bullet$ by the recipe:

$$\Gamma(X_\bullet, \mathscr{F}^\bullet) := \ker(\Gamma(X_0, \mathscr{F}^0) \to \Gamma(X_1, \mathscr{F}^1))$$

(so this is just $\Gamma(X_0, \mathscr{F}^0)$ if $X_\bullet$ is 0-truncated).

The functor $\Gamma(X_\bullet, \cdot)$ is visibly a left exact functor, and we write $\mathbf{R}\Gamma(X_\bullet, \cdot)$ to denote the resulting total derived functor (to be called *hypercohomology* on $X_\bullet$), with $\mathbf{H}^i(X_\bullet, \cdot)$ the associated hypercohomology groups.

Of course, if we're given an augmentation $a : X_\bullet \to S$ then $\Gamma(X_\bullet, \mathscr{F}^\bullet) = \Gamma(S, a_*\mathscr{F}^\bullet)$. Details on how to generate oodles of examples of cohomological descent will be given in §7. The main formal nonsense result is:

**Theorem 6.11.** *Let $X_\bullet$ be a simplicial space (without augmentation), or one which is possibly $m$-truncated for some $m \geq 0$. For any complex $K'$ in $\mathbf{D}_+(X_\bullet)$, there is a natural spectral sequence*

$$E_1^{p,q} = \mathbf{H}^q(X_p, K'|_{X_p}) \Rightarrow \mathbf{H}^{p+q}(X_\bullet, K')$$

*with $\mathrm{d}_1^{\bullet,q}$ induced by the "associated differential complex" structure along $X_\bullet$.*

*When we are given an augmentation structure $a : X_\bullet \to S$ which is of cohomological descent and we consider $K' = a^*K$ for some $K$ in $\mathbf{D}_+(S)$, then $K'|_{X_p} = a_p^*K$ and the abutment of the above spectral sequence is naturally isomorphic to $\mathbf{H}^{p+q}(S, K)$, so we get a spectral sequence*

$$E_1^{p,q} = \mathbf{H}^q(X_p, a_p^*K) \Rightarrow \mathbf{H}^{p+q}(S, K). \tag{6.3}$$

*This is all functorial in the spaces, relative to the natural pullback maps on the augmentation and simplicial complex levels.*

*Proof.* First suppose we are given an augmentation $a : X_\bullet \to S$ which is universally of cohomological descent. For $K$ in $\mathbf{D}_+(S)$, there is a canonical composite map

$$\mathbf{R}\Gamma(S, K) \to \mathbf{R}\Gamma(S, \mathbf{R}a_*(a^*K)) \simeq \mathbf{R}(\Gamma(S, \cdot) \circ a_*)(a^*K) = \mathbf{R}\Gamma(X_\bullet, a^*K)$$

with the adjunction in the first step an isomorphism because $a : X_\bullet \to S$ is *of cohomological descent* (and $\Gamma(X_\bullet, \cdot)$ is the composite of $\Gamma(S, \cdot)$ and $a_*$ by definition). This is visibly functorial in $a$ and explains the cohomological descent aspects of the assertion in the theorem in terms of the rest, for it provides maps (isomprphisms in the cohomological descent case)

$$\mathbf{H}^i(S, K) \to \mathbf{H}^i(X_\bullet, a^*K)$$

which are natural in $a$ too.

We now just have to construct the spectral sequences abutting to (hyper)cohomology on $X_\bullet$, and for this we have no need for any augmentation structure. Taking $K'$ in $\mathbf{D}_+(X_\bullet)$, we have to make a natural spectral sequence

$$E_1^{p,q} = \mathbf{H}^q(X_p, K') \Rightarrow \mathrm{R}^{p+q}\,\Gamma(X_\bullet, K')$$

with the expected $\mathrm{d}_1^{\bullet,q}$'s, and with functoriality in both $K'$ and $X_\bullet$ (and with vanishing $(p, q)$ terms when $p > m$ if $X_\bullet$ is $m$-truncated).

The idea is to compute the total derived functor $\mathbf{R}\Gamma(X_\bullet, K')$ by some other means. Let $K' \to I^\bullet$ be a quasi-isomorphism to a bounded below complex of injectives in $\mathrm{Ab}(X_\bullet)$ (so $I^q$ is an abelian sheaf on $X_\bullet$, vanishing for sufficiently negative $q$). Consider the right half-plane (nearly) first quadrant commutative diagram with $(p, q)$ term

$$\Gamma(X_p, I^q|_{X_p}). \tag{6.4}$$

In this planar diagram, the $p$th column $\Gamma(X_p, I^\bullet|_{X_p})$ is a complex in the evident manner via the differentials on $I^\bullet$, and the $q$th row is a complex via the alternating sum of pullback maps along face maps on $X_\bullet$, applied to the sheaf $I^q$ on $X_\bullet$.

The commutativity of this planar diagram follows from naturality considerations.

We create a total complex out of this in the standard manner, and we try to compute the cohomology in two ways. If we first filter by rows, then we see that the 0th column of horizontal kernels is exactly the complex

$$\Gamma(X_\bullet, I^\bullet) = \mathbf{R}\Gamma(X_\bullet, K'),$$

while we claim that all horizontal kernels away from degree 0 vanish. Once this is checked, it follows that the total complex has $n$th cohomology exactly $\mathbf{H}^n(X_\bullet, K')$, and then trying to compute using the filtration by columns will wind up giving the spectral sequence we're looking for.

In order to justify the claim just made that each row of (6.4) is acyclic away from degree 0, we claim more generally that if $I$ is an arbitrary injective in $\mathrm{Ab}(X_\bullet)$ then the complex on the $\Gamma(X_p, I|_{X_p})$'s is acyclic away from degree 0. To see this, note that the functor

$$\mathscr{F}^\bullet \rightsquigarrow \{\Gamma(X_p, \mathscr{F}^p)\}_{p \geq 0}$$

from $\mathrm{Ab}(X_\bullet)$ to $\mathrm{Ch}_{\geq 0}(\mathrm{Ab})$ has an exact left adjoint given by the term-by-term "constant sheaf" functor, it must carry injectives to injectives. But an injective object in $\mathrm{Ch}_{\geq 0}(\mathrm{Ab})$ is necessarily acyclic in positive degrees (just as is true with Ab replaced by any abelian category).

With this acyclicity verified, we pass to the column filtration on the total complex attached to our planar diagram. Now the $E_1^{p,q}$ term is obtained from forming the $q$th vertical cohomology in the $p$th column $\Gamma(X_p, I^\bullet|_{X_p})$. That is, we get a spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(\Gamma(X_p, I^\bullet|_{X_p})) \Rightarrow \mathbf{H}^{p+q}(X_\bullet, K'),$$

with the differential $\mathrm{d}_1^{\bullet,q}$ induced by the simplicial structure on the $X_p$'s. But recall that $K' \to I^\bullet$ is a quasi-isomorphism of bounded below chain complexes in $\mathrm{Ab}(X_\bullet)$, so by exactness of the restriction functors we conclude for each $p$ that $K'|_{X_p} \to I^\bullet|_{X_p}$ is a quasi-isomorphism too. Now for the magic: by Lemma 6.4, since each $I^q$ is (by construction) an injective in $\mathrm{Ab}(X_\bullet)$, its restriction $I^q|_{X_p}$ is an injective in $\mathrm{Ab}(X_p)$. Thus, $K'|_{X_p} \to I^\bullet|_{X_p}$ is a quasi-isomorphism to a bounded below complex of *injectives* in $\mathrm{Ab}(X_p)$. It follows that

$$E_1^{p,q} = \mathrm{H}^q(\Gamma(X_p, I^\bullet|_{X_p})) \simeq \mathbf{H}^q(X_p, K'|_{X_p}), \tag{6.5}$$

so we obtain the desired spectral sequence!

It is clear from this construction and from the means by which pullback maps in (hyper)cohomology are *constructed* (on the level of resolutions) that this spectral sequence is functorial in $X_\bullet$.

■

In practice, we'd also like to know that we can compute $\mathbf{H}^n(X_\bullet, K)$ for a bounded below complex $K$ in $\mathbf{D}_+(\mathrm{Ab}(X_\bullet))$ by using a quasi-isomorphism to something less esoteric than a bounded below complex of injectives (e.g., use termwise Godement or soft resolutions for ordinary topological spaces). In fact, if $K \to K'$ is a quasi-isomorphism to a bounded below complex for which the constituent terms $K'^q$ in $\mathrm{Ab}(X_\bullet)$ satisfy

$$\mathrm{H}^i(X_p, K'^q|_{X_p}) = 0$$

for all $i > 0$ and all $p, q$ (a condition which is automatic when each $K'^q$ is injective in $\mathrm{Ab}(X_\bullet)$, thanks to Lemma 6.4), then we claim that the total complex attached to the commutative planar diagram of $\Gamma(X_p, K'^q|_{X_p})$'s computes $\mathbf{R}\Gamma(X_\bullet, K)$.

Well, pick a quasi-isomorphism $K' \to I^\bullet$ to a bounded below complex of injectives. We need to show that the natural maps

$$\Gamma(X_p, K'^q|_{X_p}) \to \Gamma(X_p, I^q|_{X_p})$$

induce a quasi-isomorphism on the level of total complexes. But since each $K'^q|_{X_p}$ is $\Gamma(X_p, \cdot)$-acyclic, the $p$th column map

$$\Gamma(X_p, K'|_{X_p}) \to \Gamma(X_p, I^\bullet|_{X_p})$$

is a quasi-isomorphism since $K'|_{X_p} \to I^\bullet|_{X_p}$ is a quasi-isomorphism (!) between bounded below complexes of $\Gamma(X_p, \cdot)$-acyclics. Hence, by using the total complex filtration by columns we conclude that we get the desired quasi-isomorphism on total complexes.

Although we have constructed a nice spectral sequence whenever we're giving a morphism of cohomological descent, we have yet to provide a single example of such a morphism. The next section will show how to generate many examples. The study of criteria for cohomological descent will require a mild relativization on our spectral sequence, or rather on the aspect which doesn't require cohomological descent. Note that the following assertion includes functoriality with respect to $X_\bullet \to S$; this will be rather essential in proofs later.

**Theorem 6.12.** *If $X_\bullet$ is a simplicial space, or an $m$-truncated such object with $m \geq 0$, and $a : X_\bullet \to S$ is an augmentation with $a_p : X_p \to S$ the induced map, then for any $K$ in $\mathbf{D}_+(X_\bullet)$ there is a canonical spectral sequence*

$$E_1^{p,q} = \mathrm{R}^q\, a_{p*}(K|_{X_p}) \Rightarrow \mathrm{R}^{p+q}\, a_*(K)$$

*functorial in $a : X_\bullet \to S$.*

*Proof.* By Theorem 6.11, we have a functorial spectral sequence

$$E_1^{p,q} = \mathbf{H}^q(X_p, K|_{X_p}) \Rightarrow \mathbf{H}^{p+q}(X_\bullet, K)$$

which is also natural in $X_\bullet$. We could have just worked directly with $a_*$ and $a_{p*}$ instead of $\Gamma(X_\bullet, \cdot)$ and the $\Gamma(X_p, \cdot)$'s and redone the entire construction, essentially verbatim. This yields the desired spectral sequence. ■

## 7. Criteria for cohomological descent

We continue to work with $C$ as in the preceding section, namely a category of "spaces" with a topology defined by "étale maps".

We begin with a definition which is necessary to state the main result.

**Definition 7.1.** Let $X_\bullet$ be a simplicial space, and $a : X_\bullet \to S$ an augmentation. We say that $a$ is *universally of cohomological descent* if for every base change $S' \to S$, the augmentation $a_{/S'} : X \times_S S' \to S'$ is of cohomological descent.

We also say that a map of spaces $a_0 : X_0 \to S$ is a *map of cohomological descent* if the augmented simplicial space

$$\mathrm{cosk}_0(a_0) : \mathrm{cosk}_0(X_0/S) \to S$$

is a morphism of cohomological descent, and we say that $a_0$ is *universally of cohomological descent* if $\mathrm{cosk}_0(a_0)$ is universally of cohomological descent (i.e., $a_0$ remains a map of cohomological descent after any base change on $S$).

Before we study properties of universal cohomological descent (e.g., is it preserved under composition of maps of spaces?), we need to establish that it works in a fundamental situation which we'll see in Corollary 7.3 also provides a vast generalization of Čech theory spectral sequences.

**Theorem 7.2.** *Let $f : X \to S$ be a map of spaces which has a section locally on $S$. Then $f$ is a map universally of cohomological descent.*

*Proof.* The universality is immediate from the rest since the hypotheses are preserved under base change. We may also work locally on $S$, so we can assume $f$ has a section $\varepsilon : S \to X$. We want to show that the natural map

$$K \to \mathbf{R}a_*(a^*K)$$

is an isomorphism in $\mathbf{D}_+(S)$ for any $K$ in $\mathbf{D}_+(S)$. It suffices to take $K = \mathscr{F}$ in $\mathrm{Ab}(S)$.

We now need to recall the spectral sequence for computing $\mathbf{R}a_*(K')$ for any $K'$ in $\mathbf{D}_+(X_\bullet)$ (as in Theorem 6.12), and then we'll specialize to $K' = a^*\mathscr{F}$. For any $K'$ in $\mathbf{D}_+(X_\bullet)$, we have a spectral sequence

$$E_1^{p,q} = \mathrm{R}^q\, a_{p*}(K'|_{X_p}) \Rightarrow \mathrm{R}^{p+q}\, a_*(K')$$

in which the $q$th row $E_1^{\bullet,q}$ has differential $\mathrm{d}_1^{\bullet,q}$ given by the simplicial structure on $X_\bullet$. Note that $E_1^{\bullet,q}$ makes perfectly good sense as an augmented complex in degrees $\geq -1$, where $a_{-1}$ is the identity map on $S$ (so $E_1^{-1,q} = 0$ for $q > 0$ since pushfoward along the identity map has vanishing higher derived functors). Consider the maps

$$h_p = \varepsilon \times \mathrm{id}_{X_p} : X_p = X^{\times(p+1)} \to X^{\times(p+2)} = X_{p+1}$$

for $p \geq -1$ (where the 0th fiber power of $X$ means $S$, of course), with products taken over $S$. In the definition of $h_p$, we're simply inserting the section along the 0th coordinate of $X_{p+1}$ (e.g., $h_{-1} = \varepsilon$).

Let's see how the $h_p$'s interact with the face maps on the simplicial space $X_\bullet$ with augmentation $f : X_0 = X \to S$ (e.g., $h_{-1}$ is a section to $f$!). For any $p \geq 1$ and $0 \leq j \leq p-1$, we have

$$h_{p-1} \circ X(\partial^j_{p-1}) = X(\partial^{j+1}_p) \circ h_p \tag{7.1}$$

and $X(\partial^0_p) \circ h_p = \mathrm{id}_{X_p}$ for all $p \geq 0$.

When $K' = a^*\mathscr{F}$ with an abelian sheaf $\mathscr{F}$ on $S$, the resulting isomorphisms $h_p^*(K'|_{X_{p+1}}) \simeq K'|_{X_p}$ therefore give rise to induced pullback maps

$$E_1^{p,q} \to E_1^{p+1,q}$$

which (with the help of lots of cancellation resting on (7.1) and a bit of care in degree 0 and $-1$) form a homotopy between the identity map and the zero map on the augmented differential complex $E_1^{\bullet,q}$ (in degrees $\geq -1$). Hence, this augmented complex is acyclic! Thus, we conclude that for $q > 0$ (so $E_1^{-1,q} = 0$) the $q$th column $E_1^{\bullet,q}$ is exact (even in degree 0) whereas for $q = 0$ the column $E_1^{\bullet,0}$ with $p$th term $a_{p*}(a^*\mathscr{F}|_{X_p}) = a_{p*}(a_p^*\mathscr{F})$ (and the evident "co-simplicial" differential) is exact away from degree 0 with kernel in degree 0 given by $\mathscr{F}$ via the natural augmentation map

$$\mathscr{F} \to a_{0*}a_0^*\mathscr{F}.$$

We conclude that at the $E_2$ stage the spectral sequence collapses to just the single term $\mathscr{F}$ concentrated in the $(0,0)$ position. Hence, the total complex $\mathbf{R}a_*(a^*\mathscr{F})$ has vanishing homology in positive degrees and homology in degree 0 given by $\mathscr{F}$ *via the canonical map*. In other words, the canonical adjunction map

$$\mathscr{F} \to \mathbf{R}a_*(a^*\mathscr{F})$$

is an isomorphism for any abelian sheaf $\mathscr{F}$ on $S$ (and thus for $\mathscr{F}$ replaced by any object in $\mathbf{D}_+(S)$). This is what we needed to prove. ■

If one takes the special case of discrete topological spaces with $S = \{\emptyset\}$ a single point (so higher cohomology vanishes), the above theorem along with Theorem 6.11 yields the classical fact that any abelian group $A$ admits a resolution given by the standard combinatorial Čech construction on a non-empty set $I$. Indeed, if $I$ is viewed as discrete then $\mathrm{cosk}_0(I/\{\emptyset\})$ is an augmented simplicial space which is discrete in each degree, and we can pull back the constant sheaf $A$ from $\{\emptyset\}$. In this case $E_1^{p,q} = 0$ for all $q > 0$ while

$$E_1^{p,0} = \mathrm{Hom}_{\mathrm{Set}}(X_p, A)$$

with the usual cosimplicial Čech differential. Since $\mathrm{H}^p(\{\emptyset\}, A) = 0$ for $p > 0$, we get the desired Čech resolution of $A$ based on $I$. Actually, this is just the classical proof of exactness in disguise: to choose a section to $I \to \{\emptyset\}$ is to pick an element $i_0 \in I$, and the use of a section to make a homotopy in the proof of Theorem 7.2 then becomes exactly the classical proof of exactness of the Čech construction. See Corollary 7.12 for a hypercover generalization.

More interestingly, we can exploit our explicit knowledge of the $\mathrm{d}_1^{p,q}$ maps in the hypercovering spectral sequence in order to recover the following classical fact (note the stage of the spectral sequence and the location of the $p$, $q$ labels!):

**Corollary 7.3.** (Čech) *Suppose our site admits arbitrary coproducts, and coproducts commute with finite fiber products over a base. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an indexed cover of a space $S$, and pick an object $K$ in $\mathbf{D}_+(S)$. Then there is a spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p(\mathfrak{U}, \underline{\mathbf{H}}^q(K)) \Rightarrow \mathbf{H}^{p+q}(S, K),$$

*where $\underline{\mathbf{H}}^q(K)$ is the* presheaf *whose value on $U$ is $\mathbf{H}^q(U, K|_U)$.*

This example will be generalized to that of étale hypercovers in Corollary 7.11. An inspection of the construction shows that this spectral sequence doesn't just have the same $E_2$-terms and abutment as the classical Čech hypercohomology spectral sequence, but literally *is* the classical spectral sequence.

*Proof.* Apply Theorem 7.2 with $X = X_{\mathfrak{U}} = \coprod U_i$, with its canonical map to $S$. This satisfies the requirements of that theorem since $\{U_i\}$ is a cover of the space $S$, so we conclude that $\mathrm{cosk}_0(X_{\mathfrak{U}}/S) \to S$ is a morphism of cohomological descent. Thus, by Theorem 6.11, we get a spectral sequence

$$E_1^{p,q} = \mathbf{H}^q(X_{\mathfrak{U}}^{\times(p+1)}, K|_{X_{\mathfrak{U}}^{\times(p+1)}}) \Rightarrow \mathbf{H}^{p+q}(S, K)$$

which is natural in $K$, $\mathfrak{U}$, and $S$. Moreover, the differential on $E_1^{\bullet,q}$ is via the simplicial structure on $\mathrm{cosk}_0(X_{\mathfrak{U}}/S)$.

For fixed $q \geq 0$, we'll show

$$E_2^{p,q} = \mathrm{H}^p(E_1^{\bullet,q}) \simeq \mathrm{H}^p(\mathfrak{U}, \underline{\mathbf{H}}^q(K))$$

naturally in $K$, $\mathfrak{U}$, and $S$. This will complete the proof. The key point for the calculation is that forming cohomology of abelian sheaves on a space converts coproducts (i.e., disjoint unions) of spaces into products of abelian groups. Since $X_{\mathfrak{U}}^{\times(p+1)}$ is naturally identified with $\coprod_{\underline{i}\in I^p} U_{\underline{i}}$ where

$$U_{\underline{i}} = U_{i_0} \times_S \cdots \times_S U_{i_p}$$

for $\underline{i} = (i_0, \dots, i_p)$, we have

$$\mathbf{H}^q(X_{\mathfrak{U}}^{\times(p+1)}, K|_{X_{\mathfrak{U}}^{\times(p+1)}}) = \prod_{\underline{i}\in I^{p+1}} \mathbf{H}^q(U_{\underline{i}}, K|_{U_{\underline{i}}}) = \prod_{\underline{i}\in I^{p+1}} (\underline{\mathbf{H}}^q(K))(U_{\underline{i}}) = \mathrm{C}^p(\mathfrak{U}, \underline{\mathbf{H}}^q(K)).$$

More importantly, our description of the $\mathrm{d}_1^{p,q}$'s in terms of the simplicial structure on $X_\bullet$ shows that the differential between this term in degree $p$ and the term in degree $p+1$ is *exactly* the expected alternating sum based on the combinatorics of index-chasing in $(p+1)$-fold overlaps, etc (i.e., we get the differential from Čech theory relative to $\mathfrak{U}$). Thus, we see that the $p$th homology object $E_2^{p,q}$ on the $q$th row $E_1^{\bullet,q}$ is

$$E_2^{p,q} = \mathrm{H}^p(\mathfrak{U}, \underline{\mathbf{H}}^q(K)).$$

■

**Corollary 7.4.** *A faithfully flat scheme map $f : X_0 \to S$ which is locally of finite presentation is universally of cohomological descent relative to the étale topology.*

There is no torsion requirement on abelian sheaves for this corollary.

*Proof.* Let $a : X_\bullet \to S$ be the 0-coskeleton of $f$. By [SGA4, Cor 9.2, Exp VIII], for $\mathscr{F}$ in $\mathrm{Ab}(S)$ the natural map $\mathscr{F} \to a_* a^* \mathscr{F}$ is an isomorphism. It remains to show that $\mathrm{R}^i\, a_*(\mathscr{F}) = 0$ for $i > 0$. The problem is local for the étale topology, so we may assume $S$ is (strictly) henselian and local. For such $S$, we can use [EGA, $\mathrm{IV}_4$, 17.16.2, 18.5.11] to conclude that there exists a finite *flat* local map $g : S' \to S$ such that $X_0(S') \neq \emptyset$. In particular, $g$ is *surjective*, so the natural map $\mathscr{F} \to g_* g^* \mathscr{F}$ is injective. Thus, we can resolve $\mathscr{F}$ by a complex of sheaves which are pushfowards under $g$. Since we're trying to show that the map of triangulated functors

$$\mathrm{id} \to \mathbf{R}a_* \circ a^*$$

on $\mathbf{D}_+(S)$ is an isomorphism, we can therefore reduce to the case in which $\mathscr{F} = g_* \mathscr{F}'$ for an abelian sheaf $\mathscr{F}'$ on $S'$.

Consider the "cartesian" diagram

$$\begin{array}{ccc} X'_\bullet & \xrightarrow{a'} & S' \\ {\scriptstyle g_\bullet}\downarrow & & \downarrow{\scriptstyle g} \\ X_\bullet & \xrightarrow[a]{} & S \end{array}$$

with $g$ finite, and hence $g_\bullet$ finite in each degree. By the "finite" case of the proper base change theorem, which is valid without torsion hypotheses (and even for sheaves of sets), the natural map

$$a^* g_* \mathscr{F}' \to g_{\bullet *} a'^* \mathscr{F}'$$

is an isomorphism since this condition can be checked in each separate degree (where we have an honest cartesian diagram of schemes). We there have

$$\mathbf{R}a_*(a^*(g_*\mathscr{F}')) \simeq \mathbf{R}a_*(g_{\bullet *}(a'^*\mathscr{F}')) \simeq g_* \circ \mathbf{R}a'_*(a'^*\mathscr{F}')$$

because $a_* \circ g_{\bullet *} = g_* \circ a'_*$ with $g_*$ and $g_{\bullet *}$ both exact thanks to finiteness considerations. But the augmentation $a' : X'_\bullet \to S'$ is a 0-coskeleton for a map $X'_0 \to S'$ which has a section, so $a'$ is of cohomological descent by Theorem 7.2. Hence, $\mathbf{R}a'_*(a'^*\mathscr{F}')$ has vanishing homology sheaves in positive degrees, so the same is true for $\mathbf{R}a_*(a^*(g_*\mathscr{F}'))$. This proves that the natural map

$$g_*\mathscr{F}' \to \mathbf{R}a_*(a^* \circ g_*\mathscr{F}')$$

is an isomorphism, as the degree 0 aspect was handled for any abelian sheaf already (though it could also be deduced directly from this method, using some commutative diagram verifications).

■

We warn the reader that it is a somewhat subtle question to determine whether the property of being of cohomological descent is preserved under composition of maps of spaces. That is, if $f : Z \to T$ and $g : T \to S$ are maps of spaces which are of cohomological descent, one can ask whether $g \circ f : Z \to S$ is of cohomological descent. What this really means is to determine whether $\mathrm{cosk}_0(Z/T) \to T$ and $\mathrm{cosk}_0(T/S) \to S$ being of cohomological descent forces $\mathrm{cosk}_0(Z/S) \to S$ to be of cohomological descent. If you think for a minute, you'll see (I beleive) that this is not a tautological consequence of the definitions (even in the topological category). It might not even be true; an additional assumption of universality is needed in the proof. This preservation under composition will play a key role in Deligne's proof of the cohomological descent property for proper hypercovers. In fact, we prove more because we need more. The next result shows that morphisms universally of cohomological descent satisfy the requirements to define a Grothendieck topology.

**Theorem 7.5.** *The class of morphisms universally of cohomological descent satisfies the following properties.*

(1) *In a cartesian diagram of spaces*

$$\begin{CD} X' @>{\pi'}>> X \\ @V{f'}VV @VV{f}V \\ S' @>>{\pi}> S \end{CD}$$

*with $\pi$ universally of cohomological descent, the map $f$ is universally of cohomological descent if and only if the map $f'$ is.*

(2) *If $f : X \to Y$ and $g : Y \to Z$ are maps with $g \circ f$ universally of cohomological descent, then so is $g$.*

(3) *If $f : X \to Y$ and $g : Y \to Z$ are universally of cohomological descent, then so is $g \circ f$.*

(4) *If $f : X' \to X$ and $g : Y' \to Y$ over a base object $S$ are universally of cohomological descent, then so is $f \times g : X' \times_S Y' \to X \times_S Y$.*

*Proof.* Although we might initially be most interested in (3) and (4), in fact the hard part is (1), and both (1) and (2) are needed to prove (3) and (4).

Assuming (1) and (2), let's deduce (3) and (4). To prove (3), consider the cartesian square (with indicated section $s$):

(7.2)
$$\begin{CD} X \times_Z Y @>{h'}>> Y \\ @A{s=1\times f}AA @VV{g}V \\ @V{g'}VV \\ X @>>{h=g\circ f}> Z \end{CD}$$

Since $g$ is universally of cohomological descent, by (1) applied with the base change map $g$, it suffices to show that $h'$ is universally of cohomological descent. But $h' \circ s = f$ is universally of cohomological descent, so by (2) we conclude that $h'$ is universally of cohomological descent.

Now that (3) is known, we can deduce (4) by applying (3) to the factorization of $f \times g$ given by

$$X' \times_S Y' \overset{1\times g}{\to} X' \times_S Y \overset{f\times 1}{\to} X \times_S Y,$$

with both maps universally of cohomological descent since $g$ and $f$ are.

To show (2), we will have to assume (1). Consider the cartesian diagram (7.2), but now viewing $h$ as the base change side. Since $h$ is universally of cohomological descent, to show that $g$ is universally of cohomological descent it suffices (by (1)) to prove that $g'$ is. But $g'$ has a section, so we can use Theorem 7.2!

Finally, we turn to the most subtle part, namely (1). The implication "$\Rightarrow$" is trivial, so now assume that $f'$ is universally of cohomological descent, and we wish to deduce the same for $f$. For this we will have to make use of a doubly simplicial object, but one which may be treated without any elaborate theory. The argument which follows does the job, but it would really have been more aesthetically correct to have directly developed a general multisimplicial (or at least bisimplicial) theory from the start. Of course, Deligne uses such an approach, but I was unable to understand his proof. The following alternative argument, which should be viewed as somewhat ad hoc, is probably the same as what one would get from unwinding the very compact argument which Deligne gives.

Consider the two augmented 0-coskeleta

$$f : \mathrm{cosk}_0(X/S) \to S, \;\; \pi : \mathrm{cosk}_0(S'/S) \to S.$$

We know that $\pi$ is universally of cohomological descent and that the base change

$$f' : \mathrm{cosk}_0(X'/S') \to S'$$

of $\mathrm{cosk}_0(X/S) \to S$ by $S' \to S$ is universally of cohomological descent. It then follows that for any cartesian power $[S'/S]^{p+1} = \mathrm{cosk}_0(S'/S)_p$ of $S'$ over $S$ (with $p \geq 0$) we have that

$$\widetilde{f}_p : \mathrm{cosk}_0(X/S) \times_S \mathrm{cosk}_0(S'/S)_p \to \mathrm{cosk}_0(S'/S)_p$$

is universally of cohomological descent (as this is a base change on the case $\mathrm{cosk}_0(X'/S') \to S'$ for $p = 0$, and we note that although there are several such base change maps

$$\mathrm{cosk}_0(S'/S)_p \to S',$$

but all give the same output $\widetilde{f}_p$ since $\mathrm{cosk}_0(X/S) \to S$ starts life over the augmentation $S$ of $\mathrm{cosk}_0(S'/S)$).

Our situation is now the following. If we define

$$Z_{p,q} = \mathrm{cosk}_0(S'/S)_p \times_S \mathrm{cosk}_0(X/S)_q,$$

then $Z_{p,q}$ is naturally bisimplicial and there are augmentations

$$\widetilde{\pi}_q : Z_{\bullet,q} \to \mathrm{cosk}_0(X/S)_q$$

which are functorial in $[q]$ and hence yield an "augmentation"

$$\widetilde{\pi} : Z_{\bullet,\bullet} \to \mathrm{cosk}_0(X/S)$$

(via viewing the bisimplicial $Z_{\bullet,\bullet}$ as a simplicial object with constituent terms given by its columns). Note that for each $q \geq 0$, the row augmentation situation $\widetilde{\pi}_q$ is a base change on $\pi$ (by $\mathrm{cosk}_0(X/S)_q \to S$) and hence is universally of cohomological descent. Likewise, we get another "augmentation"

$$\widetilde{f} : Z_{\bullet,\bullet} \to \mathrm{cosk}_0(S'/S)$$

which over degree $p$ is given by the augmentation map

$$\widetilde{f}_p : Z_{p,\bullet} \to \mathrm{cosk}_0(S'/S)_p$$

which we have seen is universally of cohomological descent.

The summarizing picture is the commutative (in an evident sense) diagram:

(7.3)
$$\begin{array}{ccc} \mathrm{cosk}_0(X/S) & \xleftarrow{\widetilde{\pi}} & \{Z_{p,q} = \mathrm{cosk}_0(S'/S)_p \times_S \mathrm{cosk}_0(X/S)_q\} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ S & \xleftarrow[\pi]{} & \mathrm{cosk}_0(S'/S) \end{array}$$

in which $Z_{\bullet,\bullet}$ is bisimplicial and each of its augmented rows and columns is universally of cohomological descent (see (7.4) for a more detailed picture). Moreover, we also have that the augmentation data $\pi$ is universally of cohomological descent. We wish to infer that $f$ has the same property. To do this, we need to introduce a suitable category of "sheaves on $Z_{\bullet,\bullet}$" as well as pushfoward and pullback functors attached to $\widetilde{\pi}$ and $\widetilde{f}$ with respect to which we can meaningfully say that $\widetilde{\pi}$ and $\widetilde{f}$ are "universally of cohomological descent" (although we haven't given a development of this concept in the bisimplicial theory). Then an identity of the form

$$\widetilde{\pi}^* \circ f^* = \widetilde{f}^* \circ \pi^*$$

on *derived categories* of abelian sheaves should imply that $f^*$ is fully faithful (whence $f$ is of cohomological descent) since all other three functors are (or rather, should be) fully faithful on derived categories. Base change on $S$ would then provide the universality too. Of course, one could instead argue with (transitive) adjunction isomorphisms of derived pushfoward/pullback instead of arguing in terms of full faithfulness of derived pullback.

Let us make this strategy precise. We define a site "$Z_{\bullet,\bullet}$" in much the same spirit that we did for simplicial spaces, namely the objects are étale arrows $(U \to Z_{p,q})$, with the evident concept of morphism (now over maps in $\Delta \times \Delta$) and evident concept of cover. We thereby get a category $\widetilde{Z_{\bullet,\bullet}}$ of sheaves of sets, and a subcategory $\mathrm{Ab}(Z_{\bullet,\bullet})$ of abelian group objects. Of course, this has a very down-to-earth explication in terms of sheaves on the $Z_{p,q}$'s with oodles of maps among them, just like the naive description of $\widetilde{X_\bullet}$ in terms of sheaves on the $X_n$'s. As usual, $\mathrm{Ab}(Z_{\bullet,\bullet})$ has enough injectives.

We can define pullback and pushfoward functors on categories of sheaves of sets

$$\widetilde{\pi}_* : \widetilde{Z_{\bullet,\bullet}} \to \mathrm{cosk}_0\widetilde{(X/S)}, \;\; \widetilde{\pi}^* : \mathrm{cosk}_0\widetilde{(X/S)} \to \widetilde{Z_{\bullet,\bullet}}$$

and

$$\widetilde{f}_* : \widetilde{Z_{\bullet,\bullet}} \to \mathrm{cosk}_0\widetilde{(S'/S)}, \;\; \widetilde{f}^* : \mathrm{cosk}_0\widetilde{(S'/S)} \to \widetilde{Z_{\bullet,\bullet}}$$

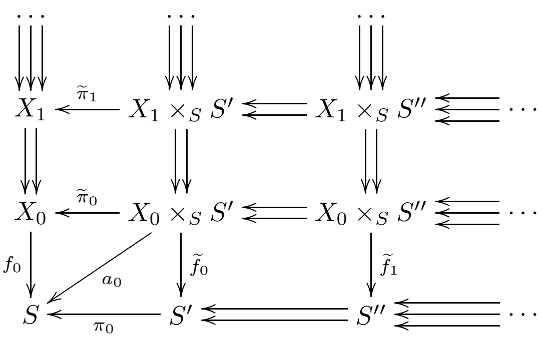
by just applying the usual such constructions in each separate row (for $\widetilde{\pi}$) and each separate column (for $\widetilde{f}$), and compatibility with the various auxiliary arrows (coming from the simplicial pullback maps all over the place) is automatic from the "commutativity" of the original geometric diagram. One checks adjointness for each pair $(\widetilde{\pi}_*, \widetilde{\pi}^*)$ and $(\widetilde{f}_*, \widetilde{f}^*)$, and we even have (easily explicated) isomorphisms of functors

$$\widetilde{\pi}^* \circ f^* \simeq \widetilde{f}^* \circ \pi^*, \;\; f_* \circ \widetilde{\pi}_* \simeq \pi_* \circ \widetilde{f}_*$$

which are moreover *compatible with adjunction* between the left sides and the right sides.

Here is a more explicit description of a corner of (7.3), in which we write $S'' := S' \times_S S'$:

(7.4)




In this diagram, for which the outer edges are "augmentation" edges while the other part is the "bisimplicial" part, we have the "bisimplicial" augmentation map

$$a_0 = f_0 \circ \widetilde{\pi}_0 = \pi_0 \circ \widetilde{f}_0 : X_0 \times_S S' \to S$$

in degree 0. Let $\mathscr{F}^{\bullet\bullet}$ be a sheaf of sets on $Z_{\bullet,\bullet}$. The common "double pushfoward"

$$f_* \widetilde{\pi}_* \mathscr{F}^{\bullet\bullet} \simeq \pi_* \widetilde{f}_* \mathscr{F}^{\bullet\bullet}$$

to $S$ is just the subsheaf of sections in $a_{0*}\mathscr{F}^{0,0}$ which, when viewed as sections of $\mathscr{F}^{0,0}$ on $X_0 \times_S S'$, have both pullbacks to $X_0 \times_S S''$ coinciding and both pullbacks to $X_1 \times_S S'$ coinciding.

We may take as an (ad hoc) *definition*

$$a_* := f_* \circ \widetilde{\pi}_* = \pi_* \circ \widetilde{f}_*$$

and

$$a^* := \widetilde{\pi}^* \circ f^* = \widetilde{f}^* \circ \pi^*.$$

These definitions are compatible with adjunction, so we also have analogous adjunction-compatible identifications on the level of total derived functors on the various derived categories of abelian sheaves. We can define what it means to say that the "morphisms" $a$, $\widetilde{f}$, or $\widetilde{\pi}$ satisfies cohomological descent: for $a$, this means either that the adjunction

$$\mathrm{id} \to \mathbf{R}a_* \circ a^*$$

is an isomorphism on $\mathbf{D}_+(S)$ or that $a^* : \mathbf{D}_+(S) \to \mathbf{D}_+(Z_{\bullet\bullet})$ is fully faithful (these two definitions being equivalent by the same adjunction argument as in the proof of Lemma 6.8). The cases of $\widetilde{f}$ and $\widetilde{\pi}$ go the same way.

Since we still have

$$a^* = \widetilde{\pi}^* \circ f^* = \widetilde{f}^* \circ \pi^*$$

on the level of derived categories, in order to deduce the full faithfulness of $f^*$ (which is what we'd like to show), it suffices to prove full faithfulness for $\widetilde{\pi}^*$ and $\widetilde{f}^*$. Indeed, once this is known then since $\pi^*$ is also fully faithful (as $\pi$ is of cohomological descent), we would get the result for $f^*$. We'd also get the *universality* aspect of the conclusion by first making a base change (as this preserves the initial hypotheses).

We now are reduced to the following general claim which treats $\widetilde{f}$ and $\widetilde{\pi}$ simultaneously. Suppose we are given a first quadrant bisimplicial space $Y_{\bullet\bullet}$ and an augmentation

$$h : Y_{\bullet\bullet} \to W_\bullet$$

to a simplicial space, say along the bottom row (we could just as well have used the left column, up to switching labelling around). We assume that for each $p \geq 0$, the induced augmented simplicial object

$$h_p : Y_{p,\bullet} \to W_p$$

is of cohomological descent. We wish to infer that the natural adjunction

$$\mathrm{id} \to \mathbf{R}h_* \circ h^* \tag{7.5}$$

is an isomorphism on $\mathbf{D}_+(W_\bullet)$. Equivalently, for $\mathscr{F}^\bullet$ in $\mathrm{Ab}(W_\bullet)$ we want that the adjunction

$$\mathscr{F}^\bullet \to h_*(h^*\mathscr{F}^\bullet) \tag{7.6}$$

is an isomorphism, and that

$$\mathrm{R}^i\, h_*(h^*\mathscr{F}^\bullet) = 0$$

for all $i > 0$.

Since a map in $\mathrm{Ab}(W_\bullet)$ is an isomorphism if and only if it is so on each $W_p$ (for $p \geq 0$), and an object vanishes in $\mathrm{Ab}(W_\bullet)$ if and only if it does on each $W_p$, it suffices to check things after restriction to each $W_p$ for $p \geq 0$. The adjunction (7.6) of pushfoward and pullback restricts to

$$\mathscr{F}^p \to h_{p*}(h_p^*\mathscr{F}^p),$$

and this is an isomorphism since $h_p$ is of cohomological descent. As for the vanishing, if we could establish that for $K$ in $\mathbf{D}_+(Z_{\bullet\bullet})$ the natural map

$$\mathbf{R}h_*(K)|_{W_p} \to \mathbf{R}h_{p*}(K|_{Z_{p,\bullet}}) \tag{7.7}$$

is an isomorphism, we would get the desired vanishing from that on the $W_p$'s (as each $h_p$ is of cohomological descent). If we think back to how we compute total direct images using an injective resolution, we just have to check that if $K \to I$ is a quasi-isomorphism to a bounded below complex of injectives in $\mathrm{Ab}(Z_{\bullet\bullet})$, then so is

$$K|_{Z_{p,\bullet}} \to I|_{Z_{p,\bullet}}$$

in $\mathrm{Ab}(Z_{p,\bullet})$. The quasi-isomorphism aspect is clear, so it suffices to check that if $I$ is an injective in $\mathrm{Ab}(Z_{\bullet\bullet})$, then $I|_{Z_{p,\bullet}}$ is an injective in $\mathrm{Ab}(Z_{p,\bullet})$ for each $p \geq 0$. This follows from Lemma 7.6 below.

■

**Lemma 7.6.** *Let $Z_{\bullet\bullet}$ be a bisimplicial space, and $I$ an injective in $\mathrm{Ab}(Z_{\bullet\bullet})$. Then $I|_{Z_{p,\bullet}}$ is an injective in $\mathrm{Ab}(Z_{p,\bullet})$ for all $p \geq 0$ and $I|_{Z_{\bullet,q}}$ is an injective in $\mathrm{Ab}(Z_{\bullet,q})$ for all $q \geq 0$.*

*Proof.* We want to just adapt the proof of Lemma 6.4 to apply in the bisimplicial context. Note one cannot just view bisimplicial objects as simplicials of simplicials and expect to just apply Lemma 6.4 directly, because the definition of the site $Z_{\bullet,\bullet}$ (or better: the category of sheaves on it) makes full use of the 2-dimensional array structure and doesn't seem easily built up from an *intrinsic* topology on a category of simplicial spaces. What the proof of Lemma 6.4 *does* readily adapt to show is that each $I|_{Z_{p,q}}$ is an injective in $\mathrm{Ab}(Z_{p,q})$. This seems not adequate, as it does not seem that spectral sequences of the sort in Theorem 6.11 can be applied (for a suitable notion of "space") to yield the $\Gamma(Z_{p,\bullet},\cdot)$-acyclicity of $I|_{Z_{p,\bullet}}$ (which would suffice). Thus, we will now show the stronger result that $I|_{Z_{p,\bullet}}$ is an injective object in $\mathrm{Ab}(Z_{p,\bullet})$ for all $p \geq 0$.

Recall the functor $L_n^{X_\bullet}$ from the proof of Lemma 6.4. Because $L_n^{X_\bullet}$ is a left adjoint to "restriction to degree $n$", it is easy to check that in the context of Lemma 6.4 the functors $L_n^{X_\bullet}$ and $L_n^{X'_\bullet}$ naturally commute with pullback with respect to a map $h : X'_\bullet \to X_\bullet$ of simplicial spaces in the sense that the natural map

$$h^*(L_n^{X_\bullet}(\mathscr{F})) \to L_n^{X'_\bullet}(h_n^*\mathscr{F})$$

is an isomorphism (one can use the construction or adjointness nonsense). Consequently, one can check that for an abelian sheaf $\mathscr{G}^\bullet$ on $Z_{p,\bullet}$, if we view $\mathscr{G}^q$ on $Z_{p,q}$ as a sheaf on the $p$th term of the simplicial object $Z_{\bullet,q}$ and apply the associated functor $L_p^{Z_{\bullet,q}}$ to this, then we get a sheaf $L_p^{Z_{\bullet,q}}(\mathscr{G}^q)$ on $Z_{\bullet,q}$ with the pullback compatibility enabling us to endow the data $\{L_p^{Z_{\bullet,q}}(\mathscr{G}^q)\}$ of sheaves on the $Z_{p,q}$'s with the structure of an object $\mathscr{L}_p(\mathscr{G}^\bullet)$ in $\mathrm{Ab}(Z_{\bullet\bullet})$. One then checks readily that this provides a left adjoint to the "restrict to $Z_{p,\bullet}$" functor from $\mathrm{Ab}(Z_{\bullet\bullet})$ to $\mathrm{Ab}(Z_{p,\bullet})$, and is *left exact* due to how it was made out of left exact functors $L_p^{Z_{\bullet,q}}$ for $q \leq 0$. This completes the proof. ■

With the basic properties of cohomological descent now established, let's think specifically about $C$ being the category of topological spaces, or perhaps schemes with the étale topology, so we have a good theory of proper maps. We will eventually prove by a coskeleton induction argument that an augmented proper hypercovering of a space $S$ is universally of cohomological descent, but in order to get the induction off the ground we need to show that a proper map of such spaces $a_0 : X_0 \to S$ is universally of cohomological descent. It is at exactly this point that the proper base change theorem plays a role (see [SGA4, Exp Vbis, pp.58–60] for an elegant simple proof of the topological proper base change theorem for proper maps between *arbitrary* topological spaces without any conditions such as local compactness or paracompactness).

**Theorem 7.7.** *Let $f : X \to S$ be a proper surjective map of topological spaces. Then $f$ is a map of cohomological descent, and remains so after base change to any other topological space $S'$. The same holds for the category of schemes with the étale topology if we work with derived categories of sheaves of $\mathbf{Z}/n$-modules for a fixed integer $n > 0$.*

*Proof.* We will write the proof using the phrase "abelian sheaf", but in the étale topology case for schemes this will be understood to mean $\mathbf{Z}/n$-module sheaves for a fixed $n > 0$.

The base change issue is clear from the rest, since properness is stable under base change. Thus, we have to just show that if $\mathscr{F}$ is an abelian sheaf on $S$ and $a : X_\bullet \to S$ is $\mathrm{cosk}_0(X_0/S)$, then $\mathscr{F} \to a_*a^*\mathscr{F}$ is an isomorphism and $\mathrm{R}^i\, a_*(a^*\mathscr{F}) = 0$ for $i > 0$. Note that all maps $a_p : X_p \to S$ are trivially proper (being just fiber powers of the proper $f$).

By Theorem 6.12, there is a spectral sequence

$$\mathrm{R}^q\, a_{p*}(K|_{X_p}) \Rightarrow \mathrm{R}^{p+q}\, a_*(K) \tag{7.8}$$

in $\mathbf{D}_+(S)$ which is natural in $a$, natural in the object $K$ in $\mathbf{D}_+(X_\bullet)$, and compatible with base change on $S$. But by the proper base change theorem, formation of higher direct images under any proper map of spaces commutes with arbitrary base change. Applying this to the $a_p$'s and using the base change compatibility of the formation of (7.8), we conclude that the more esoteric higher direct image functors $\mathrm{R}^i\, a_*$ from $\mathbf{D}_+(X_\bullet)$ to $\mathrm{Ab}(S)$ are also *of formation compatible with base change.* Applying this with $K = a^*\mathscr{F}$, we conclude that the isomorphism and vanishing questions for the $\mathrm{R}^i\, a_*(a^*\mathscr{F})$'s can be checked on the level of (geometric) fibers over $S$. That is, we may reduce to the case in which $S$ is a (geometric) point. In this case, the map $f : X \to S$ has a section (since $X$ is not empty, by the *surjectivity* of $f$), so it suffices to use Theorem 7.2. ■

Here is a nice application of Theorem 7.7 in a classical (i.e., pre-[D]) context:

*Example* 7.8. Recall the setup in Example 4.8: we have $D$ which is a strict normal crossings divisor in an ambient regular noetherian scheme or compact complex analytic space $Z$, say with irreducible components $\{D_j\}_{j\in J}$. Suppose $D$ is non-empty (so $J$ is non-empty and finite). We take $X_0 = \coprod D_j$, so there is a natural proper surjective map $X_0 \to D$. In fact, such proper surjectivity works even in the topological category with the $D_j$'s a locally finite covering of an arbitrary topological space $D$ by closed subsets (local finiteness ensures that $X_0 \to D$ is proper). We allow this as another possible initial setup. We'll now see how to use the hypercovering formalism to "compute" the cohomology of such stratified spaces (and again, in the case of the étale topology with schemes we really have to work with $\mathbf{Z}/n$-module sheaves but we'll use the more convenient phrase "abelian sheaf" anyway so as not to disrupt the discussion).

Let $X_\bullet = \operatorname{cosk}_0(X_0/D)$, so $X_\bullet$ naturally provides a proper hypercovering of $D$. By Theorem 7.9, the augmentation $X_\bullet \to D$ is universally of cohomological descent. In particular, by (6.3), for any $\mathscr{K}$ in $\mathbf{D}_+(D)$ there is a spectral sequence

$$E_1^{p,q} = \mathbf{H}^q(X_p, \mathscr{K}|_{X_p}) \Rightarrow \mathbf{H}^{p+q}(D, \mathscr{K}).$$

We want to make this a bit more explicit, and in particular cut it down to something computable. As was noted in Example 4.8, $X_p$ is just the disjoint union of $(p+1)$-fold overlaps of the $D_j$'s within $D$, but allowing for repetitions. If we define

$$D_{\underline{j}} = \bigcap_{j\in\underline{j}} D_j$$

for $\underline{j} \in J^{p+1}$, we can write the spectral sequence in the form

$$\prod_{\underline{j}\in J^{p+1}} \mathbf{H}^q(D_{\underline{j}}, \mathscr{K}|_{D_{\underline{j}}}) \Rightarrow \mathbf{H}^{p+q}(D, \mathscr{K})$$

with the evident $\mathrm{d}_1^{\bullet,q}$'s on the left side. In this spectral sequence, there's a lot of repetition due to the fact that we're essentially using an "unordered" Čech-like approach.

But just as in classical Čech theory, where we may pass to alternating cochains to get the same cohomology (which is also much more computationally useful when $J$ is finite), here too we can simplify our spectral sequence by removing redundancies and thereby bring ourselves down to a bounded spectral sequence for finite $J$. The key point is that although the general simplicial machinery is needed to functorially compute the abutment of the above spectral sequence, we see that the actual spectral sequence construction only uses the face maps of $X_\bullet$. This will enable us to make another, much smaller, spectral sequence which maps quasi-isomorphically (in an appropriate sense) to the enormous spectral sequence being considered above.

Fix an ordering on our (possibly infinite) $J$, and let $X'_n \subseteq X_n$ be the disjoint sub-union consisting of the $D_{\underline{j}}$'s for which

$$\underline{j} = (j_0, \dots, j_n) \in J^{n+1}$$

satisfies $j_0 < \cdots < j_n$. For these $\underline{j}$'s, we shall say that $D_{\underline{j}}$ "shows up" in $X'_n$. For example, $X'_{-1} = D$ and $X'_n = \emptyset$ for $n \geq |J|$. Somewhat more interestingly, in the motivating "normal crossings divisor" situation with algebraic schemes or complex analytic spaces, we see that each $D_{\underline{j}}$ which shows up in $X'_n$ has codimension $n$ in $D$ at all of its points (though $D_{\underline{j}}$ might be empty).

In general, the face maps on $X_\bullet$ carry $X'_n$ to $X'_{n-1}$ for all $n \geq 0$. We have a category $\widetilde{X'_\bullet}$ of sheaves of sets on $X'_\bullet$ defined in the evident manner, analogous to that for $X_\bullet$ (explicitly, there are pullbacks along face maps, no degeneracies), and $\mathrm{Ab}(X'_\bullet)$ has enough injectives. There is also a natural pullback functor

$$\phi^* : \widetilde{X_\bullet} \to \widetilde{X'_\bullet}.$$

Note that there is no apparent adjoint "pushfoward" map, since sheaves on $X_\bullet$ must have pullback maps along degeneracies but $X'_\bullet$ has no degeneracies. But even without such an adjoint we see that $\phi^*$ is exact since this can be checked in each degree. Moreover, we can define the functor $\Gamma(X'_\bullet, \cdot)$ on the category of sheaves of sets on $X'_\bullet$ and there is a natural transformation

$$\Gamma(X_\bullet, \cdot) \to \Gamma(X'_\bullet, \phi^*(\cdot))$$

between functors from $\mathrm{Ab}(X_\bullet)$ to Ab.

The exact same techniques used to construct the basic hypercohomology spectral sequence on $X_\bullet$ in Theorem 6.11 also work essentially verbatim on $X'_\bullet$ (we didn't need coskeleton functors or other structures which made use of degeneracies). Since $\phi^*$ is exact, it doesn't even matter if $\phi^*$ takes injective abelian sheaves to injective abelian sheaves: we still get a natural "pullback" map on the level of spectral sequences: for $K$ in $\mathbf{D}_+(X_\bullet)$ with $K' = \phi^*(K)$ in $\mathbf{D}_+(X'_\bullet)$, we get natural maps

$$E_1^{p,q}(K) = \mathbf{H}^q(X_p, K|_{X_p}) \to \mathbf{H}^q(X'_p, K'|_{X'_p}) = E_1^{p,q}(K')$$

compatible with the spectral sequence constructions on each side and with the pullback map of abutments

$$\mathbf{H}^{p+q}(X_\bullet, K) \to \mathbf{H}^{p+q}(X'_\bullet, K').$$

Now I claim that the map of row complexes

$$E_1^{\bullet,q}(K) \to E_1^{\bullet,q}(K') \tag{7.9}$$

is a quasi-isomorphism for each $q \geq 0$, whence the spectral sequences becomes isomorphic at the $E_2$-stage and the abutments must be isomorphic. Let's grant this for a moment. If $K$ is the pullback of some $\mathscr{K}$ in $\mathbf{D}_+(S)$, we can then use such a (functorial) isomorphism at the $E_2$-stage to see that the "cohomological descent" calculation of the abutment $\mathbf{H}^{p+q}(D, \mathscr{K})$ for the spectral sequence made on $X_\bullet$ also applies to give (functorially) the abutment for the sequence made on $X'_\bullet$. That is, the natural pullback map

$$\mathbf{R}\Gamma(D, \mathscr{K}) \to \mathbf{R}\Gamma(X'_\bullet, {a'}^*\mathscr{K})$$

is an isomorphism, where $a' : X'_\bullet \to S$ is the augmentation. Thus, we obtain a functorial spectral sequence

$$E_1^{p,q} = \mathbf{H}^q(X'_p, \mathscr{K}|_{X'_p}) = \prod_{j_0 < \cdots < j_p} \mathbf{H}^q(D_{\underline{j}}, \mathscr{K}|_{D_{\underline{j}}}) \Rightarrow \mathbf{H}^{p+q}(D, \mathscr{K})$$

for any $\mathscr{K}$ in $\mathbf{D}_+(D)$. Note the key input of cohomological descent took place in the context of a different spectral sequence (i.e., on the "big" $X_\bullet$, not the "small" $X'_\bullet$), and that spectral sequence just happens to give the same $E_r$-parts as the one on $X'_\bullet$ for $r \geq 2$.

It remains to establish that (7.9) is a quasi-isomorphism. In more concrete terms, in degree $p$ this map of complexes is just the natural "restriction" map

$$\mathbf{H}^q(X_p, K|_{X_p}) \to \mathbf{H}^q(X'_p, K'|_{X_p}) \tag{7.10}$$

and the differential in degree $p$ is induced by the alternating sum of pullbacks along the face maps $X_{p+1} \to X_p$ and $X'_{p+1} \to X'_p$. For any $\underline{j} = (j_0, \dots, j_p) \in J^{p+1}$ with $p \geq 0$, define the abelian group

$$A_{\underline{j}} = \mathbf{H}^q(D_{\underline{j}}, K|_{D_{\underline{j}}}).$$

Note that since we allow repetition and reorderings among the $j_i$'s, there are many different $\underline{j}$'s which give rise to the same physical $D_{\underline{j}}$. However, if we define the notation $\underline{j} \subseteq \underline{j}'$ for $\underline{j} \in J^{p+1}$ and $\underline{j}' \in J^{p'+1}$ to mean

$$\{j_0, \dots, j_p\} \subseteq \{j'_0, \dots, j'_{p'}\}$$

inside of $J$, then whenever $\underline{j} \subseteq \underline{j}'$ there is a natural "restriction" map

$$A_{\underline{j}} \to A_{\underline{j}'}$$

which is transitive in an evident sense.

We have

$$\mathbf{H}^q(X_p, K|_{X_p}) = \prod_{\underline{j} \in J^{p+1}} A_{\underline{j}}$$

and

$$\mathbf{H}^q(X'_p, K'|_{X'_p}) = \prod_{j_0 < \cdots < j_p} A_{\underline{j}},$$

with the $\mathrm{d}_1^{p,q}$'s on the hypercohomology side going over to the evident Čech-like alternating sum differentials on the right side. Moreover, under these identifications of degree $p$ terms it is clear that the map (7.10) corresponds to the projection map

$$\prod_{\underline{j} \in J^{p+1}} A_{\underline{j}} \to \prod_{j_0 < \cdots < j_p} A_{\underline{j}}$$

(which is easily seen by hand to be compatible with differentials on both sides). Now we just have to show that this map is a quasi-isomorphism of complexes. But this setup has exactly the same formal structures as in the classical setup for comparing "unordered" Čech cohomology with "ordered" Čech cohomology, so the same classical argument shows that the section map to the subcomplex of alternating cochains provides a homotopy inverse. This concludes our discussion of this example.

Now we come to the motivating result of these notes (whose proof will require a somewhat long chain of arguments, and which we'll quickly see is actually a special case of another result which is really the central theorem of these notes and has nothing to do with properness or hypercovers, but has everything to do with coskeleta). We still remain (not for long!) within the framework of ordinary spaces (i.e., topological spaces, or schemes with the étale topology), so in particular we have a good theory of properness as has been used already. The next result generalizes Theorem 7.7 from the context of 0-coskeleta to more general hypercoverings.

**Theorem 7.9.** *Let $X_\bullet \to S$ be a proper hypercovering of topological spaces. Then it is universally of cohomological descent. The same holds for the category of schemes with the étale topology, replacing abelian sheaves with sheaves of $\mathbf{Z}/n$-modules for a fixed $n > 0$.*

Note that the universality aspect is clear once the property of being of cohomological descent is proven, but in fact the mechanism of proof is to reduce to a situation where properness is not relevant but the universality is essential. It is for this reason that we state the result including the universality aspect (so as to put us in the correct frame of mind for what is about to happen). For emphasis, we will write $X_\bullet/S$ to remind ourselves of when we need to think about the entire augmented structure. As in previous discussions, we will always say "abelian sheaf" but for the étale topology on schemes it must be understood that this means $\mathbf{Z}/n$-sheaf for a fixed integer $n > 0$ (and the informed reader can then extend things to torsion sheaves and $\ell$-adic sheaves).

For a proper hypercovering $X_\bullet \to S$, each map

$$X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S))_{n+1}$$

is a proper surjection (by the definition of "proper hypercover") and hence is a map universally of cohomological descent by Theorem 7.7 (!). Thus, to prove Theorem 7.9, we are reduced to proving the following theorem which is really the key general theorem in the theory of cohomological descent. For the proof of this theorem, one again only needs to work with the extremely general concept of "space" as in §6, but thinking about more concrete examples of "space" won't impact any proofs.

**Theorem 7.10.** (Deligne) *Let $a : X_\bullet \to S$ be an augmented simplicial space with each map of spaces*

$$X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S))_{n+1}$$

*universally of cohomological descent. Then $a$ is universally of cohomological descent.*

The proof of this theorem depends crucially on the universality, and we emphasize that it does *not* involve the concept of $\mathbf{P}$-hypercovering for any easily visualized $\mathbf{P}$. Even in concrete geometric categories such as topological spaces or schemes, properness or étaleness play no role in the proof of Theorem 7.10: the *only* input of those special situations was to provide the bootstrapping device of a class of maps whose 0-coskeleta were proven to be universally of cohomological descent. In view of the stability results in Theorem 7.5, we can take $\mathbf{P}$ to be the class of morphisms universally of cohomological descent. In this case, Theorem 7.10

says that an augmented **P**-hypercovering is universally of cohomological descent (as an augmented simplicial object).

Before we launch into the proof, note that this theorem includes as a special case that of étale hypercovers of spaces (even with a very general concept of space, using the class **P** of covering morphisms), since we know by Theorem 7.2 that a covering morphism is universally of cohomological descent (as such a morphism has a section locally on the target). Thus, by Theorem 6.11 we get the following vast generalization of Čech theory (with the usual sorts of functorialities which we won't repeat again).

**Corollary 7.11.** *Let $a : X_\bullet \to S$ be an étale hypercovering of a space. Then for any $K$ in $\mathbf{D}_+(S)$, there is a spectral sequence*

$$E_1^{p,q} = \mathbf{H}^q(X_p, a_p^* K) \Rightarrow \mathbf{H}^{p+q}(S, K)$$

*in which $\mathrm{d}_1^{\bullet,q}$ is induced by the simplicial structure on $X_\bullet$.*

There's also a sheafified version of Corollary 7.11 in the spirit of Theorem 6.12 whose formulation we omit. Applying Corollary 7.11 to the case of discrete topological spaces, we get the following amusing corollary which generalizes the classical exactness for Čech complexes on an abelian group $A$ (with non-empty index set $I$) to a wider combinatorial setting where the complex is really a hypercovering of sets with "values" in $A$:

**Corollary 7.12.** *Consider the category $C$ of sets, with **P** the class of surjective maps. Let $X_\bullet/S$ be an augmented simplicial set which is a **P**-hypercovering of a point $S$ (i.e., the maps of sets $X_{n+1} \to \operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S))_{n+1}$ are surjective set maps for all $n \geq -1$). Then for any abelian group $A$, the augmented chain complex*

$$A \to \operatorname{Hom}(X_\bullet, A),$$

*with $n$th term $\operatorname{Hom}(X_n, A)$ and the evident alternating sum differential, is a resolution of $A$ (i.e., is an exact sequence of abelian groups).*

**Corollary 7.13.** *Let $a : X_\bullet \to S$ be an fppf hypercovering of a scheme $S$ (by which we really mean to use maps which are faithfully flat and locally of finite presentation). Then $a$ is universally of cohomological descent for the étale topology without any torsion restrictions.*

It seems rather difficult to construct interesting fppf hypercovers for the étale topology which are not just étale hypercovers, since the inverse limit step in the construction of coskeleta seems to generally destroy any flatness unless everything was étale over $S$ in the first place. Thus, Corollary 7.13 seems to be not very useful.

*Proof.* Combine Corollary 7.4 and Theorem 7.10. ■

Now we turn to the proof of Theorem 7.10, following Deligne's argument in [SGA4, Exp Vbis, pp.54ff]. The first key observation is that we can make the argument of inductive nature by means of the following lemma.

**Lemma 7.14.** *If $a : X_\bullet \to S$ is an augmented simplicial space for which the augmented simplicial spaces*

$$\epsilon^n : \operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S) \to S$$

*are of cohomological descent for all sufficiently large $n$, then $a$ is of cohomological descent. The same holds for universal cohomological descent.*

*Proof.* The universal case is immediate by base change, so now just suppose the maps

$$\epsilon^n : \operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S) \to S$$

are of cohomological descent for all sufficiently large $n$, and we wish to deduce that $a$ is of cohomological descent.

It suffices to prove that for any abelian sheaf $\mathscr{F}$ on $S$, we have $\mathscr{F} \simeq a_* a^* \mathscr{F}$ and $\mathrm{R}^i a_*(a^* \mathscr{F}) = 0$ for all $i > 0$. The basic idea is that by using the spectral sequence from Theorem 6.12, if we only care about $\mathrm{R}^j a_* \circ a^*$'s for $j$ below some bound, then we don't really need *all* of the data of $X_\bullet$, but only a low degree truncation. Consequently, since $\operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S)$ is the same as $X_\bullet$ in degrees $\leq n$, we'll be able to read off what we want in any degree by simply shifting attention to the $n$-coskeleton with sufficiently big $n$ (depending on what we want to study).

To be more precise, note that the statement that $\mathscr{F}$ maps isomorphically to $a_*a^*\mathscr{F}$ really only uses $\mathrm{sk}_1(X_\bullet/S)$, and we claim that the (functorial in the space!) spectral sequence

$$E_1^{p,q} = \mathrm{R}^q\, a_{p*}(a_p^*\mathscr{F}) \Rightarrow \mathrm{R}^{p+q}\, a_*(a^*\mathscr{F})$$

from Theorem 6.12 shows that $\mathrm{R}^n\, a_*(a^*\mathscr{F})$ only depends (even functorially) on $\mathrm{sk}_{2n+1} X_\bullet$ Indeed, suppose $X_\bullet \to X'_\bullet$ is a map of augmented spaces over $S$ with isomorphisms in degrees $\leq 2n+1$. For example, taking the adjunction

$$X_\bullet \to X'_\bullet = \mathrm{cosk}_{2n+1}\,\mathrm{sk}_{2n+1}(X_\bullet/S)$$

is a good example. From such a map of simplicial spaces we get maps

$$\xi_1^{p,q} : E_1^{p,q}(X'_\bullet/S) \to E_1^{p,q}(X_\bullet/S)$$

compatible with connecting maps and abutments, with $\xi_1^{p,q}$ an isomorphism whenever $p \leq 2n+1$. But $E_\infty^{p,q}$ is functorially determined by the $(p', q')$-parts of the spectral sequence with $p' + q' \leq 2(p+q)+1$. Thus, if $p + q = n$ then since we have an isomorphism on $(p', q')$-parts for all $p' \leq 2n+1$ it is automatic that the natural map

$$\mathrm{R}^n\, a'_*(a'^*\mathscr{F}) \to \mathrm{R}^n\, a_*(a^*\mathscr{F})$$

is an isomorphism.

If, for some $N \geq 1$, we apply these considerations to the natural map of simplicial spaces

$$X_\bullet \to \mathrm{cosk}_N\,\mathrm{sk}_N(X_\bullet/S)$$

which is an isomorphism in degrees $\leq N$ (over the identity map on $S$ in degree $-1$), we conclude that knowing cohomological descent for the right side ensures that $\mathscr{F} \simeq a_*a^*\mathscr{F}$ for any abelian sheaf $\mathscr{F}$ on $S$ and that $\mathrm{R}^i\, a_*(a^*\mathscr{F}) = 0$ for all $i \leq (N-1)/2$. By hypothesis we may take $N$ as large as we please, so we're done. ■

By Lemma 7.14, to prove Theorem 7.10 it suffices to prove:

**Theorem 7.15.** *If $X_\bullet \to S$ is an augmented simplicial space such that for all $-1 \leq k < n$ the maps of spaces*

$$X_{k+1} \to (\mathrm{cosk}_k\,\mathrm{sk}_k(X_\bullet/S))_{k+1}$$

*are* universally *of cohomological descent, then* $\mathrm{cosk}_n\,\mathrm{sk}_n(X_\bullet/S) \to S$ *is universally of cohomological descent.*

This will be proven by induction on $n$, and we stress the importance of the universality in the hypotheses. That is, even if we dropped universality from the conclusion, the proof would not work without universality in the hypotheses. Also, note that (with the help of the truncated variants on the skeleton and coskeleton functors; cf. Corollary 3.12) we could take $X_\bullet$ to instead be $n$-truncated without affecting the meaning of the theorem.

*Proof.* Notice that the hypothesis for $n = -1$ is empty and the conclusion for $n = -1$ is automatic. Indeed, the conclusion for $n = -1$ is that the augmentation $S_\bullet \to S$ from the constant simplicial space on $S$ is of cohomological descent, and this is Example 6.7.

The interesting case is $n \geq 0$. Let's prove the conclusion for $n = 0$ by hand to see what is going on. To treat $n = 0$ directly, we to show that the map

$$\mathrm{cosk}_0\,\mathrm{sk}_0(X_\bullet/S) \to S$$

is universally of cohomological descent. But this map is exactly the augmentation on $\mathrm{cosk}_0(X_0/S)$, so we want the map of spaces $X_0 \to S$ to be universally of cohomological descent. However, the meaning of the "$n = 0$" hypothesis (with only $k = -1$) is exactly that $X_0 \to S$ is universally of cohomological descent! Thus, we have at least been able to settle the base case $n = 0$ of the induction.

Now for the induction we pick $n \geq 0$ for which we know the theorem and we wish to prove it for $n+1$ (actually, one could have skipped the treatment of $n = 0$ and instead began the induction at $n = -1$). In particular, we have that

$$\operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S) \to S$$

is universally of cohomological descent. We wish to deduce that

$$\operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}(X_\bullet/S) \to S$$

is universally of cohomological descent. Consider the natural map

$$\operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S) \to \operatorname{cosk}_{n+1} \operatorname{sk}_{n+1} \operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S)$$

of adjunction. By Corollary 3.11 this is an isomorphism!

Thus, $Y_\bullet = \operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S)$ is an $(n+1)$-coskeleton over $S$ (i.e., is isomorphic via adjunction to its own $\operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}$) and is also known (by inductive hypothesis) to be universally of cohomological descent. Meanwhile, $Y'_\bullet = \operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}(X_\bullet/S)$ is also an $(n+1)$-coskeleton over $S$, and we can define a map

$$f : Y'_\bullet \to Y_\bullet$$

over $S$ which is just $\operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}$ applied to the canonical adjunction

$$h : X_\bullet \to \operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S).$$

Note that for $j \leq n$, $f_j : Y'_j \to Y_j$ is (up to canonical identifications) just the identity map on $X_j$ and hence is an isomorphism. Meanwhile, the map $f_{n+1}$ is naturally identified with the map of spaces

$$h_{n+1} : X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n(X_\bullet/S))_{n+1}$$

which is universally of cohomological descent by one of our initial hypotheses on the situation! Consequently, to carry out the inductive step it suffices to prove the next two general theorems which abstract the essential structure from the situation we have just formulated at the inductive step. This will complete the proof of Theorem 7.10.

■

**Theorem 7.16.** (Deligne) *Consider a map $f : Y'_\bullet/S \to Y_\bullet/S$ between two simplicial spaces. Suppose $n \geq -1$ is an integer such that*

- *The adjunctions*

$$Y_\bullet \to \operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}(Y_\bullet/S), \;\; Y'_\bullet \to \operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}(Y'_\bullet/S)$$

  *are isomorphisms.*
- *The map of spaces $f_j : Y'_j \to Y_j$ is an isomorphism for all $j \leq n$, and $f_{n+1}$ is universally of cohomological descent.*

*Then $f_m : Y'_m \to Y_m$ is universally of cohomological descent for every $m$.*

We stress that the proof of this theorem makes no use of hypercovers, properness, or étaleness. Note that the case $n = -1$ is really Theorem 7.5(3), and the proof of Theorem 7.16 uses Theorem 7.5(4) (whose justification in turn rests on knowing Theorem 7.5(3) in the first place). Thus, although we include the case $n = -1$, it's actually a case who proof really required a separate discussion earlier.

Granting Theorem 7.16 for a moment, to conclude we just need to apply the following result in which coskeletons play no explicit role, but in which there is a homotopy hypothesis which is satisfied in our situation because of Lemma 5.7 (and the fact that formation of coskeletons commutes with fiber products, thanks to adjointness reasons). It is crucial for the proof of a generalization of Theorem 7.10 (see Theorem 7.22 below) that this next result does not impose any explicit coskeleton hypothesis, even though such a condition is certainly satisfied for the cases relevant in the proof of Theorem 7.10.

**Theorem 7.17.** (Deligne) *Let $f : Y'_\bullet/S \to Y_\bullet/S$ be a map between two simplicial spaces. Assume that $f_m$ is of cohomological descent for all $m$. If $[Y'/Y]^p$ denotes the pth fiber power of $Y'_\bullet$ over $Y_\bullet$, assume also that for all $p \geq 1$ each simplicial "diagonal degeneracy" map*

$$[Y'/Y]^p \to [Y'/Y]^{p+1}$$

*is a homotopy inverse (over $S$) to both of the "projection face" maps to which it is a section.*

*If $y : Y_\bullet \to S$ and $y' : Y'_\bullet \to S$ are the augmentations, then the natural map*

$$\mathbf{R}y'_* \circ {y'}^* \to \mathbf{R}y_* \circ y^*$$

*on $\mathbf{D}_+(S)$ is an isomorphism. In particular, by transitivity of pushfoward/pullback adjunction, if $Y_\bullet/S$ is of cohomological descent then so is $Y'_\bullet/S$.*

Note the lack of a universality hypothesis. Of course, since the homotopies are required to be over $S$, if the $f_m$'s are universally of cohomological descent then all of the hypotheses are preserved by base change on $S$ and hence it follows that if $Y_\bullet/S$ is universally of cohomological descent then so is $Y'_\bullet/S$, exactly as required to complete the proof of Theorem 7.10. Although Theorem 7.17 does not seem to be explicitly stated in [SGA4], it probably is hidden somewhere in there because this lemma is the only means by which I can see how to make a proof of Theorem 7.22 (which, while not stated in [SGA4], is said to be proven there in [D]).

I am fairly certain that Theorem 7.17 is what is intended by the cryptic one-line observation near the end of the proof of [SGA4, Thm 3.3.3, Exp Vbis] (see paragraph 2, page 138 of [SGA4]) that by means of Theorem 7.16 it suffices for the proof of Theorem 7.15 (and hence Theorem 7.10) to consider the situation after base change to fiber powers. Actually, for the purpose of proving Theorem 7.10 one can get away with something whose proof is somewhat easier than Theorem 7.17. However, it seems that Theorem 7.17 is essential in the proof of the generalization Theorem 7.22 of Theorem 7.10.

Now we turn to the proofs of Theorems 7.16 and 7.17.

*Proof.* (of Theorem 7.16) As we have just noted, the case $n = -1$ is already known, so we may assume $n \geq 0$. The property of $f_m$ being universally of cohomological descent follows from the hypotheses when $m \leq n+1$. To handle $m > n+1$, we use the hypothesis that $Y_\bullet$ and $Y'_\bullet$ are actually $(n+1)$-coskeleta (over $S$). More importantly, we will really make essential use of the "universality" aspect of the hypotheses.

Fix $m > n+1$. Since $f$ is a map of simplicial objects and the adjunction

$$\mathrm{id} \to \mathrm{cosk}_{n+1} \mathrm{sk}_{n+1}$$

is *natural*, it follows from our description (better: construction) of coskeleta in terms of finite inverse limits (here working in the slice category of spaces over $S$) that the map $f_m$ is exactly the canonical map

$$\varprojlim_{\mathrm{sk}_{n+1}\Delta[m]} Y'_\phi \to \varprojlim_{\mathrm{sk}_{n+1}\Delta[m]} Y_\phi$$

induced by the various maps $f_i : Y'_i \to Y_i$ for $i \leq n+1$ (recall $Y_\phi = Y_i$ for $\phi : [i] \to [m]$ in $\Delta[m]$).

We want to make this construction of $f_m$ on inverse limits more explicit in order to see that $f_m$ is universally of cohomological descent. Recall first of all that by Corollary 3.10, to construct these inverse limits it suffices to take the limit over objects $\phi : [i] \to [m]$ (with $i \leq n+1$) such that $\phi$ is *injective*. Consider a map

(7.11)
$$\begin{array}{ccc} [i] & \xrightarrow{\alpha} & [i'] \\ & {\scriptstyle\psi}\searrow & \downarrow{\scriptstyle\psi'} \\ & & [m] \end{array}$$

in $\mathrm{sk}_{n+1}\Delta[m]$ with $\psi$ and $\psi'$ both injective. In particular, it follows that $\alpha$ must be injective! Thus, either $i < n+1$ or else $i = i' = n+1$ with $\alpha$ the identity map. This will be rather important shortly.

Let $D$ denote the full subcategory of objects $(\phi : [i] \to [m])$ in $\mathrm{sk}_{n+1}\Delta[m]$ with injective $\phi$, so the inverse limit construction $(\mathrm{cosk}_{n+1}(Y_\bullet))_m$ is given by

$$\varprojlim_D Y_\phi$$

and likewise for $Y'_\bullet$. In particular, these inverse limits can be realized inside of the respective product spaces

$$\prod_{\phi \in \mathrm{ob}(D)} Y_\phi, \quad \prod_{\phi \in \mathrm{ob}(D)} Y'_\phi$$

where we understand these products to be taken over $S$. The relations that cut out the inverse limit subobjects inside the product are exactly those imposed by maps among objects in $D$, such as in (7.11): the relation from (7.11) is that projection to the $Y_\psi$ and $Y_{\psi'}$ factors must be compatible with the map

$$Y(\alpha) : Y_{\psi'} = Y_{i'} \to Y_i = Y_\psi.$$

Let $K_\alpha \hookrightarrow \prod_{\phi \in D} Y_\phi$ be cut out by the relation (7.11), and define $K'_\alpha$ inside of $\prod_{\phi \in D} Y'_\phi$ similarly. The inverse limit $Y_m$ is just the overlap (i.e., fiber product) of the subobjects $K_\alpha$ inside of $\prod_{\phi \in D} Y_\phi$ as $\alpha$ runs over all morphisms in $D$ (as illustrated by (7.11)), and similary for making the inverse limit $Y'_m$ out of the $K'_\alpha$'s.

We have a natural map

$$\prod_{\phi \in D} Y'_\phi \to \prod_{\phi \in D} Y_\phi$$

induced by the functorial (!) map $f : Y'_\bullet \to Y_\bullet$, and this induces the map $f_m$ on inverse limits

$$\varprojlim_D Y'_\phi \to \varprojlim_D Y_\phi,$$

determined by using the map $f_i$ on $\phi$-parts for $\phi : [i] \to [m]$ an object in $D$. For any "relation" $\alpha$ as in (7.11), with $0 \leq i, i' \leq n+1$, the square

$$\begin{array}{ccc} Y'_{i'} & \xrightarrow{Y(\alpha)} & Y'_i \\ {\scriptstyle f_{i'}}\downarrow & & \downarrow{\scriptstyle f_i} \\ Y_{i'} & \xrightarrow[Y'(\alpha)]{} & Y_i \end{array}$$

is commutative because $\mathrm{sk}_{n+1}(f) : \mathrm{sk}_{n+1}(Y_\bullet/S) \to \mathrm{sk}_{n+1}(Y'_\bullet/S)$ is natural (note that $Y_i = Y_\psi$ and $Y_{i'} = Y_{\psi'}$, and similarly for $Y'_\bullet$). This ensures that the map

$$\prod_{\phi \in D} Y'_\phi \to \prod_{\phi \in D} Y_\phi$$

induced by $\mathrm{sk}_{n+1}(f)$ carries the "relation" subobject $K'_\alpha$ over into the relation subobject $K_\alpha$. Consequently, for each morphism $\alpha$ of $D$ as in (7.11) we get a commutative square

(7.12)
$$\begin{array}{ccc} K'_\alpha & \longrightarrow & \prod_{\phi \in D} Y'_\phi \\ \downarrow & & \downarrow{\scriptstyle \prod f_\phi} \\ K_\alpha & \longrightarrow & \prod_{\phi \in D} Y_\phi \end{array}$$

for a uniquely determined left column (with horizontal maps the subobject inclusions). The induced map

$$\bigcap_\alpha K'_\alpha \to \bigcap_\alpha K_\alpha$$

is exactly $f_m$, and we can view this as being induced by the various left columns in (7.12) as we vary over $\alpha$'s.

Now the miracle happens: since we managed to get ourselves to just working with the subcategory $D$ for which relations as in (7.11) only happen for either $i < n+1$ or $i = j = n+1$ with $\alpha$ the identity, we claim that the commutative squares (7.12) are *cartesian*. Indeed, for the case $i = j = n+1$ with $\alpha$ the identity, there is no non-trivial relation and hence $K_\alpha$ and $K'_\alpha$ fill up their respective product spaces. But when $i < n+1$ then

$$f_i : Y'_i = Y'_\psi \to Y_\psi = Y_i$$

is an *isomorphism* (see the initial hypotheses), so a "point"

$$y' \in \prod_{\phi \in D} Y'_\phi$$

lies in the relation $K'_\alpha$ (i.e., $Y'(\alpha)(y'_{\psi'}) = y'_\psi$ in $Y'_\psi = Y'_i$) if and only if

$$f_i(Y'(\alpha)(y'_{\psi'})) = f_i(y'_\psi)$$

in $Y_\psi = Y_i$. But

$$f_i \circ Y'(\alpha) = Y(\alpha) \circ f_{i'}$$

by the naturality of $f : Y'_\bullet \to Y_\bullet$, so the condition of membership in $K'_\alpha$ for $y'$ amounts to $f(y')$ lying in $K_\alpha$. This is exactly the statement that (7.12) is cartesian for each $\alpha$.

It then follows that the natural commutative diagram

$$\begin{array}{ccc} \varprojlim_D Y'_\phi & \longrightarrow & \prod_{\phi \in D} Y'_\phi \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle \prod f_\phi} \\ \varprojlim_D Y_\phi & \longrightarrow & \prod_{\phi \in D} Y_\phi \end{array}$$

is *cartesian.* Hence, the left column $f_m$ is universally of cohomological descent as long as the right column is of *universal* cohomological descent (we would hit a brick wall here if we didn't have the universality condition). But $f_\phi : Y'_\phi = Y'_i \to Y_i = Y_\phi$ is just $f_i$ with $i \le n+1$ and so is universally of cohomological descent for all $\phi$ in $D$ (recall the initial hypotheses!). Thus, our original claim that $f_m$ is universally of cohomological descent for all $m$ has been reduced to the assertion that if $Z'_1/S \to Z_1/S$ and $Z'_2/S \to Z_2/S$ are maps universally of cohomological descent between augmented spaces over $S$, then the product map of spaces

$$Z'_1 \times_S Z'_2 \to Z_1 \times_S Z_2$$

is universally of cohomological descent. This follows from Theorem 7.5, and completes the proof that all maps $f_m : Y'_m \to Y_m$ are universally of cohomological descent. ■

Before moving on to prove Theorem 7.17, we record an interesting consequence of the preceding argument.

**Corollary 7.18.** *Let* **P** *be a class of space morphisms stable under base change and compositions, and containing all isomorphisms, and let $n \ge 0$ be an integer. For any map $u_\bullet : X_\bullet/S \to Y_\bullet/S$ of $n$-truncated augmented simplicial objects with $u_j$ of type* **P** *for all $j \le n$, the map $\operatorname{cosk}_n(u_\bullet)$ between simplicial objects is of type* **P** *in each degree.*

*Proof.* The only issue is to check degrees $> n$. The proof of Theorem 7.16 shows that in each degree, the induced map on $n$-coskeleta can be expressed as a base change on a product of various $f_j$'s (perhaps some repeated several times). The hypotheses on **P** ensure that a map constructed in this way is also of type **P**. ■

*Proof.* (of Theorem 7.17) In order to apply the cohomological descent property of the $f_m$'s, we need to use a bisimplicial point of view similar to the proof of Theorem 7.5(1).

Consider the bisimplicial object $Z_{\bullet\bullet}$ made out of the fiber powers of $Y'_\bullet$ over $Y_\bullet$. That is,

$$Z_{p,q} = Y'_p \times_{Y_p} \cdots \times_{Y_p} Y'_p = \operatorname{cosk}_0(Y'_p/Y_p)_q$$

is a $(q+1)$-fold fiber product of $Y'_p$ over $Y_p$ (with $p, q \ge 0$), with $p$th column

$$Z_{p,\bullet} = \operatorname{cosk}_0(Y'_p/Y_p)$$

as a simplicial object (so we have an augmentation to $Y_\bullet/S$ along the bottom edge induced by the map $f$ from $Y'_\bullet = Z_{\bullet,0}$ to $Y_\bullet$) and $q$th row $Z_{\bullet,q} = [Y'/Y]^{q+1}$ given by the $(q+1)$th fiber power of $Y'_\bullet$ over $Y_\bullet$. We also insert an augmentation to the constant simplicial space $S_\bullet/S$ along the left edge, so the summarizing picture is:

$$\begin{array}{ccc} \mathrm{cosk}_0(S/S) & \xleftarrow{\widetilde{\pi}} & \{Z_{p,q} = \mathrm{cosk}_0(Y'_p/Y_p)_q\} \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\widetilde{f}} \\ S & \xleftarrow[y]{} & Y_\bullet \end{array} \tag{7.13}$$

with augmentation $\widetilde{\pi}_0$ along the 0th row exactly $y' : Y'_\bullet \to S$, and $\widetilde{f}$ essentially the original map $f : Y'_\bullet \to Y_\bullet$ (up to whether we want to focus on it as a map from the 0th row $Z_{\bullet,0}$ to $Y_\bullet$ or as a pair of pushfoward/pullback functors (i.e. map of topoi) between the categories of sheaves of sets on $Z_{\bullet\bullet}$ and $Y_\bullet$). Of course, the notions of sheaves of sets on $Z_{\bullet\bullet}$ and functors such as

$$\widetilde{f}_* : \widetilde{Z_{\bullet\bullet}} \to Y_\bullet, \ \ \widetilde{\pi}^* : \widetilde{S_\bullet} \to \widetilde{Z_{\bullet\bullet}}$$

and so on are defined just as in the proof of Theorem 7.5(1).

We introduce the map

$$a_0 = \varepsilon \circ \widetilde{\pi}_0 = y_0 \circ f_0 : Z_{0,0} = Y'_0 \to S$$

(so this is just $y'_0 : Y'_0 \to S$) and the pair of adjoint functors $a_*$ and $a^*$ between $\widetilde{Z_{\bullet\bullet}}$ and $\widetilde{S}$ just as in the proof of Theorem 7.5(1). Since the augmented columns of $Z_{\bullet\bullet}$ are of cohomological descent, we conclude just as in the proof of Theorem 7.5(1) (see (7.5)) that $\widetilde{f}$ is of cohomological descent in the sense that

$$\mathrm{id} \to \mathbf{R}\widetilde{f}_* \circ \widetilde{f}^*$$

is an isomorphism on $\mathbf{D}_+(Y_\bullet)$, or equivalently that

$$\widetilde{f}^* : \mathbf{D}_+(Y_\bullet) \to \mathbf{D}_+(Z_{\bullet\bullet})$$

is fully faithful. Thus, we conclude that the natural map

$$\mathbf{R}y_* \circ y^* \to \mathbf{R}y_* \circ \mathbf{R}\widetilde{f}_* \circ \widetilde{f}^* \circ y^* = \mathbf{R}a_* \circ a^* \tag{7.14}$$

is an isomorphism on $\mathbf{D}_+(S)$.

We now factor the "map" $a : Z_{\bullet\bullet} \to S$ (really viewed as an adjoint pair of functors) as $a = \varepsilon \circ \widetilde{\pi}$, so using (7.14) yields a composite isomorphism

$$\mathbf{R}y_* \circ y^* \simeq \mathbf{R}a_* \circ a^* \simeq \mathbf{R}\varepsilon_* \circ (\mathbf{R}\widetilde{\pi}_* \circ \widetilde{\pi}^*) \circ \varepsilon^*.$$

The fact that $\widetilde{\pi}$ along the 0th row is exactly $y' : Y'_\bullet \to S$ is what will provide the link with $y'$ which will eventually yield that the natural map

$$\mathbf{R}y_* \circ y^* \to \mathbf{R}y'_* \circ {y'}^* \tag{7.15}$$

is an isomorphism (which is what we want to prove). In fact, we're going to make a degenerate spectral sequence which will yield an isomorphism

$$\mathbf{R}a_* \circ a^* \simeq \mathbf{R}\widetilde{\pi}_{0*} \circ \widetilde{\pi}_0^* = \mathbf{R}y'_* \circ {y'}^*$$

whose composite with (7.14) will be exactly (7.15).

Using the Dold-Kan correspondence

$$\mathrm{Ab}(S_\bullet) = \mathrm{Cosimp}(\mathrm{Ab}(S)) \simeq \mathrm{Ch}_{\geq 0}(\mathrm{Ab}(S)),$$

the functor $\varepsilon_*$ on $\mathrm{Ab}(S_\bullet)$ is identified with the functor $\underline{\mathrm{H}}^0$ on $\mathrm{Ch}_{\geq 0}(\mathrm{Ab}(S))$ which extracts the 0th homology. Thus, its derived functors $\mathrm{R}^j\varepsilon_*$ are just the $\underline{\mathrm{H}}^j$'s as a $\delta$-functor. But recall that under the Dold-Kan correspondence, for any abelian cosimplicial sheaf $A^\bullet$ on $S$, the homology of the associated "normalized" chain complex (which is what sets up the Dold-Kan correspondence) is *$\delta$-functorially* isomorphic to the homology of the chain complex $\mathbf{s}(A^\bullet)$ whose $n$th term is $A^n$ and whose $n$th differential is the alternating sum of face maps $A^n \to A^{n+1}$. In other words, if $\mathscr{F}^\bullet$ is an object in $\mathrm{Ab}(S_\bullet)$, then $\delta$-functorially

$$\mathrm{R}^j\varepsilon_*(\mathscr{F}^\bullet) \simeq \underline{\mathrm{H}}^j(\mathbf{s}(\mathscr{F}^\bullet)),$$

and we'll just write $\underline{\mathrm{H}}^j(\mathscr{F}^\bullet)$ to denote this homology sheaf (with the evident $\delta$-functor structure, though the $\delta$-functorial compatibility will not be relevant in what follows). Thus, we have a natural Grothendieck-Leray spectral sequence

$$E_2^{p,q} = \underline{\mathrm{H}}^p(\mathrm{R}^q\,\widetilde{\pi}_*(\mathscr{G}^{\bullet\bullet})) \Rightarrow \mathrm{R}^{p+q}\,a_*(\mathscr{G}^{\bullet\bullet})$$

for any $\mathscr{G}^{\bullet\bullet}$ in $\mathrm{Ab}(Z_{\bullet\bullet})$. In particular, for any $\mathscr{F}$ in $\mathrm{Ab}(S)$ there is a natural spectral sequence

$$E_2^{p,q} = \underline{\mathrm{H}}^p(\mathrm{R}^q\,\widetilde{\pi}_*(a^*\mathscr{F})) \Rightarrow \mathrm{R}^{p+q}\,a_*(a^*\mathscr{F}). \tag{7.16}$$

We are going to prove that (7.16) degenerates at the $E_2$ stage, with $E_2^{p,q} = 0$ for $p > 0$ and

$$E_2^{0,q} \simeq \mathrm{R}^q\,\widetilde{\pi}_{0*}(\widetilde{\pi}_0^*\mathscr{F}) = \mathrm{R}^q\,y'_*(y'^*\mathscr{F}),$$

and this will essentially give us what we wanted to prove (modulo some compatibility verifications). Pick an injective resolution $I^\bullet$ of $a^*\mathscr{F}$ in $\mathrm{Ab}(Z_{\bullet\bullet})$, so each $I^r$ is a sheaf on $Z_{\bullet\bullet}$. In order to keep the notation clear, we will write $I_r^{\bullet\bullet}$ instead of $I_r$. For example,

$$I_0^{\bullet,q} \to I_1^{\bullet,q} \to \dots$$

is a resolution of $\widetilde{\pi}_q^*(\varepsilon^*\mathscr{F}) = a^*\mathscr{F}|_{Z_{\bullet,q}}$ in $\mathrm{Ab}(Z_{\bullet,q})$, and is even an *injective* resolution, for the restriction functor

$$\mathrm{Ab}(Z_{\bullet\bullet}) \to \mathrm{Ab}(Z_{\bullet,q})$$

carries injectives (such as $I_r = I_r^{\bullet\bullet}$) to injectives, by Lemma 7.6. Of course, the object $\mathrm{R}^j\,\widetilde{\pi}_*(a^*\mathscr{F})$ in $\mathrm{Ab}(S_\bullet)$ is the $j$th homology of the complex

$$\widetilde{\pi}_*(I_0) \to \widetilde{\pi}_*(I_1) \to \dots$$

in $\mathrm{Ab}(S_\bullet)$.

In order to get a better handle on this situation, let $I = I_r$ for a fixed $r \geq 0$, and consider the cosimplicial sheaf $\widetilde{\pi}_*(I)$ on $S$. Actually, for clarity let's consider this pushfoward for $I$ an arbitrary abelian sheaf on $Z_{\bullet\bullet}$. The pushfoward $\widetilde{\pi}_*(I)$ is a cosimplicial sheaf on $S$ with $q$th term $\widetilde{\pi}_{q*}(I^{\bullet,q})$ and cosimiplicial structure induced by the vertical face and degeneracy maps in $Z_{\bullet\bullet}$ (all of which are maps over $S$). Consider a face or degeneracy map between two adjacent rows $Z_{\bullet,q}$ and $Z_{\bullet,q+1}$ with $q \geq 0$. This may be viewed as a map between two augmented simplicial spaces over $S$, with augmentations

$$\widetilde{\pi}_q : Z_{\bullet,q} \to S,\ \ \widetilde{\pi}_{q+1} : Z_{\bullet,q+1} \to S.$$

We now apply Lemma 5.6 to the category $C'$ of objects in the site $Z_{\bullet\bullet}$ (in which the physical object $Z_{\bullet\bullet}$ makes sense as a bisimplicial object, with each row and column a simplicial object). More specifically, if we consider the rows $Z_{\bullet,q}$ and $Z_{\bullet,q+1}$ as simplicial objects in $C'$, then by the main homotopy hypothesis of the theorem we're trying to prove, each "diagonal degeneracy" map

$$\sigma_q^j : Z_{\bullet,q} \to Z_{\bullet,q+1}$$

for $0 \leq j \leq q$ is a simplicial map which is a homotopy inverse to the two "projection face" maps $\partial_{q+1}^j$ and $\partial_{q+1}^{j+1}$ to which it is a section. Taking $F$ in Lemma 5.6 to be the contravariant functor which assigns to any object $(U \to Z_{p,q})$ in $C'$ the pushfoward to $S$ of the restriction $I^{p,q}|_U$, we conclude that the resulting "total pushfowards" $F(Z_{\bullet,q})$ and $F(Z_{\bullet,q+1})$ as cosimplicial abelian sheaves on $S$ have the remarkable property that the map

$$F(\sigma) : F(Z_{\bullet,q+1}) \to F(Z_{\bullet,q})$$

induced by any vertical degeneracy

$$\sigma = \sigma_q^j : Z_{\bullet,q} \to Z_{\bullet,q+1}$$

is a homotopy inverse to the map induced by both vertical faces

$$\partial_{q+1}^j, \partial_{q+1}^{j+1} : Z_{\bullet,q+1} \to Z_{\bullet,q}$$

to which $\sigma$ is a section, where "homotopy inverse" may be understood to mean on the level of chain complexes $\mathbf{s}(F(Z_{\bullet,q}))$ and $\mathbf{s}(F(Z_{\bullet,q+1}))$.

Thus, if we pass to the 0th homologies $\widetilde{\pi}_{q*}(I|_{Z_{\bullet,q}})$ and $\widetilde{\pi}_{q+1\,*}(I|_{Z_{\bullet,q+1}})$ of these chain complexes, it follows that each map

$$s_j^q : \widetilde{\pi}_{q+1\,*}(I|_{Z_{\bullet,q+1}}) \to \widetilde{\pi}_{q*}(I|_{Z_{\bullet,q}})$$

in $\mathrm{Ab}(S_\bullet)$ induced by a vertical degeneracy $\sigma_q^j$ is a genuine *inverse* to the map induced by each of the vertical faces $\partial_{q+1}^j, \partial_{q+1}^{j+1}$. Thus, when we consider $\widetilde{\pi}_*(I)$ as a cosimplicial sheaf on $S$, any two "consecutive" face maps $\mathrm{d}_j$ and $\mathrm{d}_{j+1}$ from degree $q$ to degree $q+1$ are *equal* to a common isomorphism (namely, the inverse of a certain degeneracy map). Now varying $j$ shows that *all* the face maps in degree $q$ are equal to a common isomorphism. Consequently, if we pass to the chain complex

$$\mathbf{s}(\widetilde{\pi}_*(I))$$

in $\mathrm{Ab}(S)$ whose differentials are alternating sums of face maps (a sum over $0 \le j \le n+1$ in degree $n$), it follows that the differentials in even degree are 0 and the differentials in odd degree are isomorphisms. Thus, the homology of this chain complex vanishes in positive degrees and is isomorphic to $\widetilde{\pi}_{0*}(I|_{Z_{\bullet,0}})$ in degree 0!!

Now consider the computation of

$$\mathrm{R}^q\,\widetilde{\pi}_*(\mathscr{G}^{\bullet\bullet})$$

as a cosimplicial sheaf on $S$ for an abelian sheaf $\mathscr{G}^{\bullet\bullet}$ on $Z_{\bullet\bullet}$ (we'll specialize to $\mathscr{G}^{\bullet\bullet}$ arising as pullback from $S_\bullet$, or even $S$, shortly). We pick an injective resolution

$$I_0^{\bullet\bullet} \to I_1^{\bullet\bullet} \to \dots$$

of $\mathscr{G}^{\bullet\bullet}$, and then apply $\widetilde{\pi}_*$ to get a complex

$$\widetilde{\pi}_*(I_0^{\bullet\bullet}) \to \widetilde{\pi}_*(I_1^{\bullet\bullet}) \to \dots$$

in $\mathrm{Ab}(S_\bullet)$ whose $q$th homology in $\mathrm{Ab}(S_\bullet)$ is exactly $\mathrm{R}^q\,\widetilde{\pi}_*(\mathscr{G}^{\bullet\bullet})$. The preceding calculation shows that each $\widetilde{\pi}_*(I_r^{\bullet\bullet})$ in $\mathrm{Ab}(S_\bullet)$ is sent by

$$\mathrm{Ab}(S_\bullet) \simeq \mathrm{Cosimp}(\mathrm{Ab}(S)) \xrightarrow{\mathbf{s}} \mathrm{Ch}_{\ge 0}(\mathrm{Ab}(S))$$

to a chain complex with vanishing differentials in even degrees and isomorphism differentials in odd degrees. This property is therefore inherited by the homology sheaves $\mathrm{R}^q\,\widetilde{\pi}_*(\mathscr{G}^{\bullet\bullet})$ on $S_\bullet$. Thus, applying $\mathrm{R}^p\,\varepsilon_* = \underline{\mathrm{H}}^p \circ \mathbf{s}$ to $\mathrm{R}^q\,\widetilde{\pi}_*(\mathscr{G}^{\bullet\bullet})$ yields 0 when $p > 0$ and yields

$$\mathrm{R}^q\,\widetilde{\pi}_*(\mathscr{G}^{\bullet\bullet})|_{S_0} \simeq \mathrm{R}^q\,\widetilde{\pi}_{0*}(\mathscr{G}^{\bullet\bullet}|_{Z_{\bullet,0}})$$

when $p = 0$, where this latter isomorphism of course uses the fact that restricting an injective in $\mathrm{Ab}(Z_{\bullet\bullet})$ to a fixed row or column (e.g., $Z_{\bullet,0}$) yields an injective abelian sheaf on that row or column.

Since $Z_{\bullet,0} = Y'_\bullet$ with augmentation $\widetilde{\pi}_0 = y'$, by taking $\mathscr{G}^{\bullet\bullet} = a^*(\mathscr{F})$ for an abelian sheaf $\mathscr{F}$ on $S$ (so $\mathscr{G}^{\bullet\bullet}|_{Z_{\bullet,0}}$ is just $y'^*\mathscr{F}$) we conclude that

$$\mathrm{R}^p\,\varepsilon_*(\mathrm{R}^q\,\widetilde{\pi}_*(a^*\mathscr{F})) = \begin{cases} 0, & p > 0 \\ \mathrm{R}^q\,y'_*(y'^*\mathscr{F}), & p = 0 \end{cases}$$

In other words, in the natural spectral sequence (7.16) for computing $\mathrm{R}^n\,a_*(a^*\mathscr{F})$, we have $E_2^{p,q} = 0$ when $p > 0$ and degeneration yields a natural edge isomorphism

$$\mathrm{R}^n\,y'_*(y'^*\mathscr{F}) \simeq \mathrm{R}^n\,a_*(a^*\mathscr{F}).$$

It remains to check that for all $n \ge 0$, the diagram

$$\begin{array}{ccc} \mathrm{R}^n\,y_*(y^*\mathscr{F}) & \longrightarrow & \mathrm{R}^n\,y'_*(y'^*\mathscr{F}) \\ & \underset{\simeq}{\searrow} & \big\downarrow \simeq \\ & & \mathrm{R}^n\,a_*(a^*\mathscr{F}) \end{array} \tag{7.17}$$

commutes, where the top row is "pullback" along $f : Y'_\bullet/S \to Y_\bullet/S$, the right vertical arrow is the edge map just shown to be an isomorphism, and the diagonal arrow is "pullback" along the augmentation

$$\widetilde{f} : Z_{\bullet\bullet} \to Y_\bullet.$$

One way to check the commutativity is to identify all three sides with $\mathrm{H}^n$'s of natural adjunction maps on the level of derived categories, but a simpler way (in the present context, where we're evaluating on the level of abelian sheaves) is to just chase how everthing is constructed in terms of injective resolutions of abelian sheaves. ■

Let us conclude these notes by discussing a mild "relativization" of the theory (also due to Deligne, but apparently not mentioned in [SGA4]). So far, we have studied the phenomenon of cohomological descent for augmented simplicial objects

$$a : X_\bullet \to S.$$

We also saw in the discussion preceding Definition 6.5 that such an $a$ can be uniquely factored as $a = \varepsilon_S \circ a_\bullet$ where $\varepsilon_S : S_\bullet \to S$ is the constant simplicial object on $S$ and

$$a_\bullet : X_\bullet \to S_\bullet$$

is a map respecting augmentations. It is reasonable to ask if we can somewhat enlarge the scope of the preceding conclusions by focusing more on $a_\bullet$ instead of the augmentation. More specifically, suppose

$$X_\bullet \to S, \;\; Y_\bullet \to S$$

are two augmented simplicial objects over $S$, and

$$u_\bullet : X_\bullet \to Y_\bullet$$

is a map respecting the augmentations. We can ask if there is a good theory of cohomological descent for $u_\bullet$ which will recover the earlier theory when we take $Y_\bullet/S = S_\bullet/S$ to be the constant augmented object.

The basic setup goes as follows. We fix a category $C$ which has finite fiber products, and a fix an augmented simplicial object $Y_\bullet/S$ in $C$. Consider the functor on slice categories

$$\mathrm{sk}_n^{Y_\bullet} : \mathrm{Simp}(C/S)/(Y_\bullet/S) \to \mathrm{Simp}_n(C/S)/\,\mathrm{sk}_n(Y_\bullet/S)$$

which assigns to any $X_\bullet/S \to Y_\bullet/S$ the truncation $\mathrm{sk}_n(X_\bullet/S) \to \mathrm{sk}_n(Y_\bullet/S)$ for $n \geq -1$. Taking $Y_\bullet/S = S_\bullet/S$ recovers the old skeleton functors for simplicial objects in the slice category $C/S$. In any case, one can ask if this functor has a right adjoint, and it does:

$$\mathrm{cosk}_n^{Y_\bullet}(Z_\bullet/S) = \mathrm{cosk}_n(Z_\bullet/S) \times_{\mathrm{cosk}_n \mathrm{sk}_n(Y_\bullet/S)} (Y_\bullet/S).$$

The functor $\mathrm{sk}_{-1}$ has constant value $S$ and the adjoint $\mathrm{cosk}_{-1}^{Y_\bullet}$ sends the unique object $S$ to $Y_\bullet/S$.

By using the easy fact that these modified skeleton and (right adjoint!) coskeleton functors still commute with finite inverse limits (such as fiber products), one readily checks (by reduction to the first part of Corollary 3.11) that the adjunction

$$\mathrm{sk}_n^{Y_\bullet} \circ \mathrm{cosk}_n^{Y_\bullet} \to \mathrm{id}$$

is an isomorphism. Also, the basic properties of coskeleta (such as Corollary 3.11) remain true in this more general setting. Note, however, that there seems to be no analogue of Theorem 6.12 for $u_\bullet$.

The basic homotopy property of coskeleta in Lemma 5.7 also carries over:

**Lemma 7.19.** *Let $T_\bullet/S$ be an augmented simplicial object in $C$, and let $f, g : X'_\bullet/S \to X_\bullet/S$ be maps of $n$-truncated augmented simplicial objects in $C$ over $\mathrm{sk}_n(T_\bullet/S)$, where $n \geq 0$ is an integer. Assume that $f$ and $g$ coincide on $(n-1)$-skeleta (a vacuous condition if $n = 0$). Then there exists a strict simplicial homotopy (over $S$) between $\mathrm{cosk}_n^{T_\bullet}(f)$ and $\mathrm{cosk}_n^{T_\bullet}(g)$.*

*In particular, if $f : X'_\bullet/S \to X_\bullet/S$ and $s : X_\bullet/S \to X'_\bullet/S$ are maps of $n$-truncated augmented simplicial spaces over $\mathrm{sk}_n(T_\bullet/S)$ with $f$ and $s$ inverse on $(n-1)$-skeleta and $s_n$ a section to $f_n$, then $\mathrm{cosk}_n^{T_\bullet}(f)$ and $\mathrm{cosk}_n^{T_\bullet}(s)$ are strict homotopy inverse to each other.*

*Proof.* By Lemma 5.7, $\mathrm{cosk}_n(f)$ and $\mathrm{cosk}_n(g)$ are strictly homotopic (these coskeleta being computed relative to the augmentation to $S$, of course). Recall that by construction

$$\mathrm{cosk}_n^{T_\bullet}(X_\bullet/S) = \mathrm{cosk}_n(X_\bullet/S) \times_{\mathrm{cosk}_n \mathrm{sk}_n(T_\bullet/S)} (T_\bullet/S),$$

and similarly for $X'_\bullet$. We must check that the strict homotopy constructed between $\mathrm{cosk}_n(f)$ and $\mathrm{cosk}_n(g)$ in the proof of Lemma 5.7 is well-behaved with respect to base change by the adjunction

$$T_\bullet/S \to \mathrm{cosk}_n\,\mathrm{sk}_n(T_\bullet/S).$$

Since the diagram

$$\begin{array}{ccc} X_\bullet/S & \overset{f}{\underset{g}{\rightrightarrows}} & Y_\bullet/S \\ \downarrow & & \downarrow \\ \mathrm{sk}_n(T_\bullet/S) & \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} & \mathrm{sk}_n(T_\bullet/S) \end{array}$$

commutes, we can apply the naturality aspect of Lemma 5.7, keeping in mind that when Lemma 5.7 is applied to two equal maps $h : Z'_\bullet/S \to Z_\bullet/S$, such as in the bottom row of the preceding diagram, the resulting homotopy is the trivial one (corresponding to the map $\Delta[1]_i \to \mathrm{Hom}_C(Z'_i, Z_i)$ sending everything to $h_i$, which in the case of a category admitting finite coproducts corresponds to the projection $\Delta[1] \times Z'_\bullet \to Z_\bullet$ defined by $h$). Thus, the homotopy $H$ between the two maps

$$\mathrm{cosk}_n(f), \mathrm{cosk}_n(g) : \mathrm{cosk}_n(X_\bullet/S) \to \mathrm{cosk}_n(Y_\bullet/S)$$

is compatible with the "identity" homotopy on $\mathrm{cosk}_n\,\mathrm{sk}_n(T_\bullet/S)$.

In other words, the map

$$H_i : \Delta[1]_i \to \mathrm{Hom}_C(\mathrm{cosk}_n(X_\bullet/S)_i, \mathrm{cosk}_n(Y_\bullet/S)_i)$$

has values in $C$-maps over $(\mathrm{cosk}_n\,\mathrm{sk}_n(T_\bullet/S))_i$, and the naturality of $X_\bullet/S \to \mathrm{sk}_n(T_\bullet/S)$ and $Y_\bullet/S \to \mathrm{sk}_n(T_\bullet/S)$ and the functoriality of $\mathrm{cosk}_n$ ensure that everything here respects functoriality in $[i]$. A more picturesque description of this situation can be given when $C$ admits finite coproducts: it says that the diagram

$$\begin{array}{ccc} \Delta[1] \times \mathrm{cosk}_n(X_\bullet/S) & \xrightarrow{H} & \mathrm{cosk}_n(Y_\bullet/S) \\ \downarrow & & \downarrow \\ \Delta[1] \times \mathrm{cosk}_n\,\mathrm{sk}_n(T_\bullet/S) & \xrightarrow[\mathrm{pr}_2]{} & \mathrm{cosk}_n\,\mathrm{sk}_n(T_\bullet/S) \end{array}$$

commutes.

In any case, we may make the base change by the maps

$$T_i \to (\mathrm{cosk}_n\,\mathrm{sk}_n(T_\bullet/S))_i$$

which respect functoriality in $[i]$, and this yields maps

$$H'_i : \Delta[1]_i \to \mathrm{Hom}_{C/T_i}(\mathrm{cosk}_n^{T_\bullet}(X_\bullet/S)_i, \mathrm{cosk}_n^{T_\bullet}(Y_\bullet/S)_i)$$

which carry $(\iota_0)_i$ to $\mathrm{cosk}_n^{T_\bullet}(f)_i$ and $(\iota_1)_i$ to $\mathrm{cosk}_n^{T_\bullet}(g)_i$, and respect naturality in $[i]$. The reader can easily supply a more visual description when $C$ admits finite coproducts, and either way we see that this data $H'$ is exactly the desired homotopy between $\mathrm{cosk}_n^{T_\bullet}(f)$ and $\mathrm{cosk}_n^{T_\bullet}(g)$ (and is even readily checked to enjoy a naturality property analogous to that shown in Lemma 5.7 for "ordinary" coskeleta over $S$). ∎

We now make a definition which, for $Y_\bullet/S = S_\bullet/S$, recovers our earlier notion of **P**-hypercovering (originally formulated only for "maps" of the type $a : X_\bullet \to S$):

**Definition 7.20.** Let **P** be a class of morphisms in $C$ which is stable under base change and composition, and contains all isomorphisms. We say that a map

$$u_\bullet : X_\bullet \to Y_\bullet$$

of augmented objects over $S$ is a **P**-*hypercovering* if

$$X_{n+1} \to (\mathrm{cosk}_n^{Y_\bullet}\,\mathrm{sk}_n^{Y_\bullet}(X_\bullet/S))_{n+1} \simeq (\mathrm{cosk}_n\,\mathrm{sk}_n(X_\bullet/S))_{n+1} \times_{(\mathrm{cosk}_n\,\mathrm{sk}_n(Y_\bullet/S))_{n+1}} Y_{n+1}$$

is in **P** for each $n \geq -1$.

*Example* 7.21. When $Y_\bullet/S = S_\bullet/S$, this coincides with the earlier notion of **P**-hypercovering. When $n = -1$, then since

$$\operatorname{cosk}_{-1}^{Y_\bullet} \operatorname{sk}_{-1}^{Y_\bullet}(X_\bullet/S) = Y_\bullet/S$$

we see that the condition is that $X_0 \to Y_0$ be of type **P**.

Passing to the case of simplicial *spaces*, we can now speak of $u_\bullet$ being a proper hypercovering, or an étale hypercovering, or *more generally* a hypercovering relative to the class of morphisms which are universally of cohomological descent (note that Theorem 7.5 ensures that this class of morphisms satisfies the stability requirements for **P** as in Definition 7.20). The natural extension of Theorem 7.10 is:

**Theorem 7.22.** *Consider $u_\bullet : X_\bullet/S \to T_\bullet/S$ a map of augmented simplicial spaces, and let*

$$x : X_\bullet \to S, \;\; t : T_\bullet \to S$$

*be the augmentations. If $u_\bullet$ is a hypercovering with respect to the class of morphisms universally of cohomological descent, then the natural "pullback" map*

$$\mathbf{R}t_* \circ t^* \to \mathbf{R}t_* \circ \mathbf{R}u_{\bullet *} \circ u_\bullet^* \circ t^* \simeq \mathbf{R}x_* \circ x^* \tag{7.18}$$

*on $\mathbf{D}_+(S)$ is an isomorphism.*

When $T_\bullet/S = S_\bullet/S$, this is exactly Theorem 7.10 (thanks to Example 6.7 and some easy compatibility checks).

*Proof.* For ease of exposition, let's refer to the condition of (7.18) being an isomorphism as the property that $X_\bullet$ (or more accurately, $u_\bullet$) is *of $T_\bullet/S$-cohomological descent.* The proof that $X_\bullet$ is of $T_\bullet/S$-cohomological descent whenever $u_\bullet$ is a "universal cohomological descent" hypercovering basically amounts to modifying the string of arguments that proved Theorem 7.10.

It suffices to show that for $\mathscr{F}$ in $\mathrm{Ab}(S)$, the map

$$\mathbf{R}t_*(t^*\mathscr{F}) \to \mathbf{R}x_*(x^*\mathscr{F}) \tag{7.19}$$

is an isomorphism. By the "functoriality in spaces" aspect of Theorem 6.12 (applied to $u_\bullet$), we have a commutative diagram of spectral sequences

$$\begin{array}{ccc} \mathrm{R}^q\, t_{p*}(t_p^*\mathscr{F}) & \Rightarrow & \mathrm{R}^{p+q}\, t_*(t^*\mathscr{F}) \\ \downarrow & & \downarrow \\ \mathrm{R}^q\, x_{p*}(x_p^*\mathscr{F}) & \Rightarrow & \mathrm{R}^{p+q}\, x_*(x^*\mathscr{F}). \end{array}$$

Thus, we can use the technique of proof of Lemma 7.14 to conclude that the property of (7.19) being an isomorphism on $n$th homology only depends on $\operatorname{sk}_{2n+1}(u_\bullet)$. Thus, by using the adjunction maps

$$X_\bullet \to \operatorname{cosk}_N^{T_\bullet} \operatorname{sk}_N^{T_\bullet}(X_\bullet/S)$$

over $T_\bullet/S$ for all $N$, we see that if $\operatorname{cosk}_N^{T_\bullet} \operatorname{sk}_N^{T_\bullet}(X_\bullet/S)$ is of $T_\bullet/S$-cohomological descent for all $N$ then so is $X_\bullet$, as desired.

Recalling the hypothesis on our original $u_\bullet$, we can therefore reduce to the analogue of Theorem 7.15: if $n \geq -1$ is an integer and $u_\bullet : X_\bullet/S \to T_\bullet/S$ is a map of augmented simplicial spaces such that for all $-1 \leq k < n$ the degree $k+1$ map

$$X_{k+1} \to (\operatorname{cosk}_k^{T_\bullet} \operatorname{sk}_k^{T_\bullet}(X_\bullet/S))_{k+1}$$

is universally of cohomological descent, then we claim that $\operatorname{cosk}_n^{T_\bullet} \operatorname{sk}_n^{T_\bullet}(X_\bullet/S)$ is of $T_\bullet/S$-cohomological descent. The case $n = -1$ is trivial, since

$$\operatorname{cosk}_{-1}^{T_\bullet} \operatorname{sk}_{-1}^{T_\bullet}(X_\bullet/S) = T_\bullet.$$

On the other hand, the case $n = 0$ (so just $k = -1$) is not so trivial, and this contrasts with the setup in the proof of Theorem 7.15, where we could readily verify the case $n = 0$. The hypothesis for $n = 0$ amounts to requiring that $X_0 \to T_0$ be universally of cohomological descent, and we wish to conclude that

$$\operatorname{cosk}_0^{T_\bullet}(X_0/S) = \operatorname{cosk}_0(X_0/S) \times_{\operatorname{cosk}_0(T_0/S)} (T_\bullet/S)$$

is of $T_\bullet/S$-cohomological descent. Even when $T_\bullet = \operatorname{cosk}_0(T_0/S)$ this is not obvious (to me): this special case says that if $X_0 \to T_0$ is a map over $S$ which is universally of cohomological descent, then $\operatorname{cosk}_0(X_0/S)$ is of $\operatorname{cosk}_0(T_0/S)/S$-cohomological descent. Thus, we really do begin the induction at the trivial case $n = -1$.

Since the analogue of Corollary 3.11 for the functors $\operatorname{sk}_n^{T_\bullet}$ and $\operatorname{cosk}_n^{T_\bullet}$ is true, we can use the exact same inductive argument as in the proof of Theorem 7.15 (beginning the induction at the trivial case $n = -1$). We use the functors $\operatorname{cosk}_n^{T_\bullet}$ and $\operatorname{sk}_n^{T_\bullet}$, and consider $T_\bullet/S$-cohomological descent, whereas in the proof of Theorem 7.15 we only considered (universal) cohomological descent over $S$ (or what comes to the same, $S_\bullet/S$-cohomological descent).

By letting

$$Y'_\bullet = \operatorname{cosk}_{n+1}^{T_\bullet} \operatorname{sk}_{n+1}^{T_\bullet}(X_\bullet/S),\ \ Y_\bullet = \operatorname{cosk}_n^{T_\bullet} \operatorname{sk}_n^{T_\bullet}(X_\bullet/S)$$

we have a natural map

$$f : Y'_\bullet/S \to Y_\bullet/S \tag{7.20}$$

obtained by applying $\operatorname{cosk}_{n+1}^{T_\bullet} \operatorname{sk}_{n+1}^{T_\bullet}$ to the canonical adjunction

$$h : X_\bullet \to \operatorname{cosk}_n^{T_\bullet} \operatorname{sk}_n^{T_\bullet}(X_\bullet/S),$$

and by a transitivity argument it suffices to show that $Y'_\bullet$ is of $Y_\bullet$-cohomological descent (for our arbitrary but fixed $n \geq -1$ which we have suppressed from the notation $Y_\bullet$ and $Y'_\bullet$).

As in the proof of Theorem 7.15, the map $f_j$ is an isomorphism for $j \leq n$ while $f_{n+1}$ is identified with the map

$$X_{n+1} \to (\operatorname{cosk}_n^{T_\bullet} \operatorname{sk}_n^{T_\bullet}(X_\bullet/S))_{n+1}$$

which *by hypothesis on* $u_\bullet : X_\bullet/S \to T_\bullet/S$ is universally of cohomological descent. We want to apply Theorem 7.17 to (7.20), and this requires we check that all maps $f_m$ are of cohomological descent and that the homotopy hypothesis in Theorem 7.17 is satisfied (it is important here that Theorem 7.17 does not require that anything be a coskeleton relative to $S$). We will obtain the (universal) cohomological descent property for the $f_m$'s by an indirect application of Theorem 7.16 (rather than have to rework the proof of Theorem 7.16 for more general skeleton and coskeleton functors), and then we'll check the homotopy requirement.

We argue as follows. Since $f_j$ is an isomorphism for all $j \leq n$ and $f_{n+1}$ is universally of cohomological descent, by applying $\operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}$ throughout (functors relative to $S_\bullet/S$) we conclude via Theorem 7.16 that the simplicial map $\operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}(f)$ induces a degreewise map of spaces which (for each degree) is universally of cohomological descent. Thus, for all $m > n+1$ the *base change* map

$$(\operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}(Y'_\bullet/S))_m \times_{(\operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}(T_\bullet/S))_m} T_m \to (\operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}(Y_\bullet/S))_m \times_{(\operatorname{cosk}_{n+1} \operatorname{sk}_{n+1}(T_\bullet/S))_m} T_m$$

is also (universally) of cohomological descent. But this map is exactly the map $\operatorname{cosk}_{n+1}^{T_\bullet} \operatorname{sk}_{n+1}^{T_\bullet}(f)$, in degree $m$. Since $f$ is a map between $\operatorname{cosk}_{n+1}^{T_\bullet}$'s, we may make the identification

$$\operatorname{cosk}_{n+1}^{T_\bullet} \operatorname{sk}_{n+1}^{T_\bullet}(f) = f.$$

Thus, we conclude that for all $m > n+1$, the map of spaces $f_m$ is (universally) of cohomological descent. That is, $f_m$ is (universally) of cohomological descent for all $m > n+1$. Hence, $f_j$ is (universally) of cohomological descent for every $j$.

To apply Theorem 7.17, it remains to check that for all $p \geq 1$ and all $0 \leq j < p$, the simplicial fiber power "face" projections

$$\partial^j, \partial^{j+1} : [Y'/Y]^{p+1} \to [Y'/Y]^p$$

are homotopy inverse to the common diagonal "degeneracy" section $\sigma^j$, with these all viewed as maps of simplicial spaces. Since $\operatorname{sk}_n^{T_\bullet}$ and $\operatorname{cosk}_n^{T_\bullet}$ commute with fiber products, we may invoke Lemma 7.19 (replacing the use of Lemma 5.7 in the proof of Theorem 7.10). ■

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