18.034 PROBLEM SET 2

Due February 18 in class. No lates will be accepted. Discussion is encouraged, with two caveats: (a) write up your solutions by yourself, and (b) give credit when others came up with ideas (you won't be penalized for this). Give explanations, not just answers. References are to Boyce and DiPrima; the answers to problems from the book are in the back of the book.

1. (p. 97 # 43) A pond forms as water collects in a conical depression of radius a and depth h. Suppose that water flows in at a constant rate k, and is lost through evaporation at a rate proportional to the surface area.
   (a) Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation
   $$dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3},$$
   where α is the coefficient of evaporation.
   (b) Find the equilibrium depth of water in the pond. Is the equilibrium stable?
   (c) Find a condition that must be satisfied if the pond is not to overflow.
2. (p. 32 # 35) **Discontinuous coefficients.** Solve the initial value problem $y' + 2y = g(t)$, $y(0) = 0$, where $g(t)$ takes on the value 1 when $0 \leq t \leq 1$, and 0 when $t > 1$. (There is a little deceit going on here; really you should ignore the value $t = 1$. You may want to think about this.)
3. **Exact equations.**
   (a) Solve the following differential equation $(e^x \sin y - 2y \sin x) + (e^x \cos y + 2 \cos x)dy/dx = 0$, with the initial value $y(1) = 1$.
   (b) (p. 89 # 13) Solve $(2x - y) + (2y - x)dy/dx = 0$ with the initial condition $y(1) = 3$, and determine at least approximately where the solution is valid (i.e. what range $\alpha < x < \beta$).
   (c) (p. 89 # 16) Find the value of b for which the given equation is exact, and then solve it using that value of b: $(ye^{2xy} + x) + bxe^{2xy}dy/dx = 0$.
   (d) (p. 89 # 24) Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation $M + Ny' = 0$ has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.
4. **Bernoulli equations.** (p. 33) To solve a differential equations of the form
   $$y' + p(t)y = q(t)y^n$$
   (a *Bernoulli equation*) where $n \neq 0, 1$, make the substitution $v = y^{1-n}$ to turn it into a linear equation (in v, not y). (You should remember this technique.) Use this method to solve:
   (a) $t^2y' + 2ty - 4y^3 = 0$, $t > 0$,
   (b) (p. 33 # 40) $y' = \epsilon y - \sigma y^3$, ϵ, $\sigma > 0$.

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5. **Riccati equations.** (a) (p. 95, # 33) The equation

$$y' = q_1(t) + q_2(t)y + q_3(t)y^2$$

is known as a Riccati equation. Suppose that some particular solution y_1 of this equation is known. A more general solution containing one arbitrary constant can be obtained through the substitution $y = y_1(t) + 1/v(t)$. Show that $v(t)$ satisfies the first-order *linear* equation $v' = -(q_2 + 2q_3y_1)v - q_3$. Note that $v(t)$ will contain a singular arbitrary constant.
(b) (p. 96 # 34 (a), (b)) Using the method of (a) and the given particular solution, solve each of the following Riccati equations:

$$y' = 1 + t^2 - 2ty + y^2; y_1(t) = t,$$
$$y' = -1/t^2 - y/t + y^2; y_1(t) = 1/t.$$

6. **Homogeneous equations.** Read Section 2.9. (*Know* this technique!) In each of the following problems, show that the given equation is homogeneous. Solve the differential equation. Draw a direction field and some integral curves.
(a) (p. 93 # 8) $(x^2 + 3xy + y^2) - x^2\frac{dy}{dx} = 0$.
(b) (p. 93 # 10)

$$\frac{dy}{dx} = \frac{y^4 + 2xy^3 - 3x^2y^2 - 2x^3y}{2x^2y^2 - 2x^3y - 2x^4}.$$

7. Consider the differential equation

$$\left(\frac{-y}{x^2+y^2} - \frac{-y}{(x-1)^2+y^2}\right) + \left(\frac{x}{x^2+y^2} - \frac{x-1}{(x-1)^2+y^2}\right)\frac{dy}{dx} = 0.$$

Notice that the coefficients are continuous on the plane minus the points $A = (0, 0)$ and $B = (1, 0)$. (To better understand where this comes from, consider the function $\tan^{-1}(y/(x-1))$.) For which of the following regions is the differential equation exact, and why? (To save you some computational hassle: the equation is closed, i.e. if you write it as $M + Ny' = 0$, then $M_y = N_t$.)