18.034 FINAL EXAM PRACTICE PROBLEMS

The following ten problems are intended as practice for the final exam. These problems are more skewed toward the end of the course than the final. For problems related to earlier parts of the course, see the practice problems for the midterms. For more information on the final, see the course webpage.

Problem 1. Consider the differential equation

$$M(x,y) + N(x,y)y' = 0, \tag{1}$$

where M, N, M_y, N_x are continuous on the entire plane.

(a) What does it mean for equation (1) to be exact? What does this tell you about solutions to the the differential equation?

(b) What is the condition required on M and N such that equation (1) is exact?

(c) If equation (1) is not exact, then it may be made so by multiplying by an integrating factor $\mu(x)$ (a function only of x). Show that this integrating factor will make (1) exact if

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$

Hence this method will only work if $\frac{M_y - N_x}{N}$ is a function of x only.

(d) Apply this technique to show that any solution to the differential equation $(3xy + y^2) + (x^2 + xy)y' = 0$ lies on a curve of the form $x^3y + x^2y^2/2 = \text{constant}$.

Problem 2. Consider the differential equation

$$t^2y'' - (t+2)ty' + (t+2)y = 0.$$

Notice that $y_1 = t$ is a solution. Find all solutions on the interval $t > 0$. Hint: Suppose you had another solution y_2. Find the Wronskian, using Abel's theorem, and from this construct a differential equation satisfied by y_2.

Problem 3. Consider the differential equation

$$xy'' + 2y' + xy = 0.$$

(a) Which values of x are ordinary points? Regular singular points? (Hint: there is one; call it x_0.) Irregular singular points (i.e. neither ordinary nor regular singular)?

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(b) Solve the *indicial equation* at the regular singular point x_0 to get the *exponents of the singularity* r_1, r_2.

(c) Find two linearly independent solutions of the form

$$(x - x_0)^{r_i} \sum_{n=0}^{\infty} a_n x^n.$$

(d)* Show that if $y(-\pi) \neq 0$, then as t approaches 0 from the left (i.e. from the negative side), $y(t)$ becomes (either positively or negatively) infinite. (Hint: Express the solutions from (c) in terms of elementary functions.)

Problem 4. Consider the differential equation coming from a door with friction

$$y'' + \gamma y' + y = 0.$$

Here γ corresponds to the friction; assume it is small but positive. Show that as γ increases, the period of oscillation increases.

Problem 5. Show that t^3 and t^4 can't both be solutions to a differential equation of the form $y'' + qy' + ry = 0$ where q and r are continuous functions defined on the real numbers. Can t^3 and t^4 be solutions to a differential equation of the form $py'' + qy' + ry = 0$ where p, q and r are continuous functions defined on the real numbers?

Problem 6. Solve the system of equations

$$x' = 4x + y,$$
$$y' = -x + 2y.$$

Sketch the phase portrait.

Problem 7. Consider the system of differential equations

$$x' = -x - y,$$
$$y' = x - y.$$

(a)What are the eigenvectors and eigenvalues?

(b) Sketch the phase portrait.

(c) Find a fundamental matrix $\Psi(t)$ for the system. If

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

(d) Show that $e^{2\pi A} = \begin{pmatrix} e^{-2\pi} & 0 \\ 0 & e^{-2\pi} \end{pmatrix}$, where e^{At} is defined to be $\Psi(t)\Psi(0)^{-1}$.

Problem 8. (a) Find the coefficient of x^n in the power series expansion of $(1 + x)^r$, where r is a real number.

(b) Show that $e^{ax}e^{bx} = e^{(a+b)x}$ by explicitly multiplying out the power series expansions of both sides.

(c) Show that $2\sin(x)\cos(x) = \sin(2x)$ by explicitly multiplying out the power series expansions of both sides.

Problem 9. consider the differential equation

$$(x-1)^2 y'' - (x-1)y' + 2y = 0. \tag{2}$$

(a) What is a regular singular point? Show that 1 is a regular singular point of equation (2).

(b) Find all solutions away from $x = 1$.

(c) Sketch a non-zero solution of equation (2) near $x = 1$.

(d) Why do solutions of $(x-1)^2 y'' - (x-1)y' + (2 + (x-1)^4)y = 0$ have similar behavior?

Problem 10. (a) By the method of variation of parameters show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t),$$

$y(0) = 0$, $y'(0) = 0$ is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t-\tau) d\tau.$$

(b) Show that if $f(t) = \delta(t-\tau)$, then the solution of part (a) reduces to

$$y = u_\pi(t) e^{-(t-\pi)} \sin(t-\pi).$$

Sketch this solution.

(c) Use the Laplace transform to solve the given initial value problem with $f(t) = \delta(t-\pi)$ and confirm that the solution agrees with the result of part (b).