18.034 MIDTERM 2

Explain your answers clearly; show all steps. Calculators may not be used. All problems have equal value. Please put your name on every sheet. Good luck!

1. (a) y_1, y_2, and y_3 are 3 solutions of the differential equation $(1-t)y''' + y'' + t^2y' + t^3y = 0$ on the interval $1 < t < \infty$. Calculate the function $W(y_1, y_2, y_3)(t)$ given that $W(y_1, y_2, y_3)(2) = 3$.

(b) The equation $y' + a(x)y = 0$ has for a solution

$$\phi(x) = e^{-\int_{x_0}^{x} a(t)dt}.$$

(Here let a be continuous on an interval I containing x_0.) This suggests trying to find a solution of

$$L(y) = y'' + a_1(x)y' + a_2(x)y = 0$$

of the form

$$\phi(x) = e^{\int_{x_0}^{x} p(t)dt}$$

where p is a function to be determined. Show that ϕ is a solution of $L(y) = 0$ if and only if p satisfies the first-order non-linear equation $y' = -y^2 - a_1(x)y - a_2(x)$. (Remark: This last equation is called a *Riccati equation*.)

2. (a) Consider the equation $y'' - \frac{2}{x^2}y = 0$ (for $0 < x < \infty$). Find all solutions. (*Hint:* Try functions of the form $y = x^r$. How do you know you've found *all* the solutions?)

(b) Find all solutions to the equation $y'' - \frac{2}{x^2}y = x$. *Hint:* Use "variation of parameters". Suppose ϕ_1 and ϕ_2 are linearly independent solutions to the homogeneous version of the equation (see (a)). Look for a solution of the form $\phi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$.

3. Iterate $x \to \sqrt{1+x}$. Start with $x = 0$. What happens?

4. (a) State the Existence and Uniqueness Theorem for differential equations of the form $y' = f(x, y)$.

(b) Consider the differential equation $y' = t^2(y+1)$ on the interval $\mathbb{R}$, with initial condition $y(0) = 0$. Find a solution $y = \phi(t)$ defined for all $t \in \mathbb{R}$. If the first few Picard iterates (used in the proof of the Existence and Uniqueness Theorem described in (a)) are $\phi_0(t) = 0$, $\phi_1(t)$, $\phi_2(t)$, find $\phi_1(t)$ and $\phi_2(t)$.

(c) Explain why the $\phi_1(t)$ and $\phi_2(t)$ you found are approximations to $\phi(t)$.

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5. Consider the equation $y'' + \cos(x)y' + \sin(x)y = 0$.

(a) Let $\phi(x)$ be a nontrivial solution, and let $\psi(x) = \phi(x + 2\pi)$. Prove that $\psi(x)$ is also a solution.

(b) Show that $\phi(x)$ is a periodic solution of period 2π if, and only if, $\phi(0) = \phi(2\pi)$ and $\phi'(0) = \phi'(2\pi)$.

(c) Let $\phi_1(x)$, $\phi_2(x)$ be two solutions satisfying $\phi_1(0) = 1$, $\phi_1'(0) = 0$, $\phi_2(0) = 0$, $\phi_2'(0) = 1$. Show that there are constants a and b such that

$$\phi_1(x + 2\pi) = a\phi_1(x) + b\phi_2(x).$$

(*Hint:* See (a).)

6. Let $A = \begin{pmatrix} 3 & 1 \\ -5 & -3 \end{pmatrix}$. Find the eigenvalues of A. Find eigenvectors of A corresponding to each of the eigenvalues. Calculate A^{2000}.