# 1 The Haar functions and the Brownian motion

## 1.1 The Haar functions and their completeness

### The Haar functions

The basic Haar function is

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2, \\ -1 & \text{if } 1/2 \le x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

It has mean zero

$$\int_0^1 \psi(x)dx = 0,$$

and is normalized so that

$$\int_0^1 \psi^2(x)dx = 1.$$

The rescaled and shifted Haar functions are

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}.$$

These functions form an orthonormal set in $L^2(\mathbb{R})$ because if $j = j'$ and $k \neq k'$ then

$$\int_{\mathbb{R}} \psi_{jk}(x)\psi_{jk'}(x)dx = 2^j \int_{\mathbb{R}} \psi(2^j x - k)\psi(2^j x - k')dx = 0$$

because $\psi(y-k)\psi(y-k') = 0$ for any $y \in \mathbb{R}$ and $k \neq k'$. On the other hand, if $j \neq j'$, say, $j < j'$, then

$$\begin{aligned}
&\int_{\mathbb{R}} \psi_{jk}(x)\psi_{j'k'}(x)dx = 2^{j/2+j'/2}\int_{\mathbb{R}} \psi(2^j x - k)\psi(2^{j'} x - k')dx \\
&= 2^{j'/2-j/2}\int_{\mathbb{R}} \psi(y)\psi(2^{j'-j}y + 2^{j'-j}k - k')dy \\
&= 2^{j'/2-j/2}\int_0^{1/2} \psi(2^{j'-j}y + 2^{j'-j}k - k')dy - 2^{j'/2-j/2}\int_{1/2}^1 \psi(2^{j'-j}y + 2^{j'-j}k - k')dy.
\end{aligned}$$

Both of the integrals above equal to zero. Indeed, $2^{j'-j} \ge 2$, hence, for instance,

$$\int_0^{1/2} \psi(2^{j'-j}y + 2^{j'-j}k - k')dy = 2^{j-j'}\int_0^{2^{j'-j-1}} \psi(y + 2^{j'-j}k - k')dy = 0,$$

because

$$\int_m^n \psi(y)dy = 0,$$

for all $m, n \in \mathbb{Z}$, and $j' > j$. Finally, when $j = j'$, $k = k'$ we have

$$\int_{\mathbb{R}} |\psi_{jk}(x)|^2 = 2^j \int_{\mathbb{R}} |\psi(2^j x - k)|^2 dx = \int_{\mathbb{R}} |\psi(x-k)|^2 dx = 1.$$

The Haar coefficients of a function $f \in L^2(\mathbb{R})$ are defined as the inner products

$$c_{jk} = \int f(x)\psi_{jk}(x)dx, \tag{1.2}$$

and the Haar series of $f$ is

$$\sum_{j,k\in\mathbb{Z}} c_{jk}\psi_{jk}(x). \tag{1.3}$$

Orthonormality of the family $\{\psi_{jk}\}$ ensures that

$$\sum_{j,k} |c_{jk}|^2 \leq \|f\|_{L^2}^2 < +\infty,$$

and the series (1.3) converges in $L^2(\mathbb{R})$. In order to show that it actually converges to the function $f$ itself we need to prove that the Haar functions form a basis for $L^2(\mathbb{R})$.

### Completeness of the Haar functions

To show that Haar functions form a basis in $L^2(\mathbb{R})$ we consider the dyadic projections $P_n$ defined as follows. Given $f \in L^2(\mathbb{R})$, and $n, k \in \mathbb{Z}$, consider the intervals

$$I_{nk} = ((k-1)/2^n, k/2^n],$$

then

$$P_n f(x) = -\!\!\!\!\!\!\int_{I_{nk}} f dx = 2^n \int_{I_{nk}} f dx, \quad \text{for } x \in I_{nk}.$$

The function $P_n f$ is constant on each of the dyadic intervals $I_{nk}$. In particular, each Haar function $\psi_{jk}$ satisfies $P_n\psi_{jk}(x) = 0$ for $j \geq n$, while $P_n\psi_{jk}(x) = \psi_{jk}(x)$ for $j < n$. We claim that, actually, for any $f \in L^2(\mathbb{R})$ we have

$$P_{n+1}f - P_n f = \sum_{k\in\mathbb{Z}} c_{nk}\psi_{nk}(x), \tag{1.4}$$

with the Haar coefficients $c_{nk}$ given by (1.2). Indeed, let $x \in I_{nk}$ and write

$$I_{nk} = \Big(\frac{(k-1)}{2^n}, \frac{k}{2^n}\Big] = \Big(\frac{2(k-1)}{2^{n+1}}, \frac{(2k-1)}{2^{n+1}}\Big] \bigcup \Big(\frac{(2k-1)}{2^{n+1}}, \frac{2k}{2^{n+1}}\Big] = I_{n+1,2k-1} \bigcup I_{n+1,2k}.$$

The function $P_n f$ is constant on the whole interval $I_{nk}$ while $P_{n+1}f$ is constant on each of the sub-intervals $I_{n+1,2k-1}$ and $I_{n+1,2k}$. In addition,

$$\int_{I_{nk}} (P_n f)dx = \int_{I_{nk}} (P_{n+1}f)dx.$$

This means exactly that

$$P_{n+1}(x) = P_n f(x) + c_{nk}\psi_{nk}(x) \text{ for } x \in I_{nk},$$

which is (1.4).

As a consequence of (1.4) we deduce that

$$P_{n+1}f(x) - P_{-m}f(x) = \sum_{j=-m}^{n} \sum_{k\in\mathbb{Z}} c_{jk}\psi_{jk}(x), \tag{1.5}$$

for all $m, n \in \mathbb{Z}$ with $n > m$. It remains to show that for any $f \in L^2(\mathbb{R})$ we have

$$\lim_{m\to+\infty} P_{-m}f(x) = 0, \qquad \lim_{n\to+\infty} P_n f(x) = f(x), \tag{1.6}$$

both in the $L^2$-sense. The operators $P_n f$ are uniformly bounded because for all $n, k \in \mathbb{Z}$ we have

$$\int_{I_{nk}} |(P_n f)(x)|^2 dx = 2^{-n}2^{2n} \left| \int_{I_{nk}} f(y)dy \right|^2 \le \int_{I_{nk}} |f(y)|^2 dy.$$

Summing over $k \in \mathbb{Z}$ for a fixed $n$ we get

$$\int_{\mathbb{R}} |P_n f(x)|^2 \le \int_{\mathbb{R}} |f(x)|^2,$$

thus $\|P_n f\|_{L^2} \le \|f\|_{L^2}$. Uniform boundedness of $P_n$ implies that it is sufficient to establish both limits in (1.6) for functions $f \in C_c(\mathbb{R})$. However, for such $f$ we have, on one hand,

$$|P_{-m}f(x)| \le \frac{1}{2^m} \int_{\mathbb{R}} |f(x)|dx \to 0 \text{ as } m \to +\infty,$$

and, on the other, $f$ is uniformly continuous on $\mathbb{R}$, so that $\|P_n f(x) - f(x)\|_{L^\infty} \to 0$ as $n \to +\infty$, which, as both $P_n f$ and $f$ are compactly supported, implies the second limit in (1.6). Therefore, $\psi_{jk}$ form an orthonormal basis in $L^2(\mathbb{R})$ and every function $f \in L^2(\mathbb{R})$ has the reperesentation

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{jk}\psi_{jk}(x), \quad c_{jk} = \int_{\mathbb{R}} f(y)\psi_{jk}(y)dy. \tag{1.7}$$

## 1.2 The Brownian motion

Brownian motion is a random process $X_t(\omega)$, $t \ge 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which has the following properties:

(i) The function $X_t(\omega)$ is continuous in $t$ for a.e. realization $\omega$.

(ii) For all $0 \le s < t < +\infty$ the random variable $X_t(\omega) - X_s(\omega)$ is Gaussian with mean zero and variance $t - s$:

$$\mathbb{E}(X(t) - X(s)) = 0, \quad \mathbb{E}(X(t) - X(s))^2 = t - s.$$

(iii) For any subdivision $0 = t_0 < t_1 < \ldots < t_N = t$ of the interval $[0, t]$, the random variables $X_{t_1} - X_{t_0}, \ldots, X_{t_N} - X_{t_{N-1}}$ are independent.

## Construction of the Brownian motion

We will construct the Brownian motion on the interval $0 \le t \le 1$ – the restriction to a finite interval is a simple convenience but by no means a necessity. The Haar functions $\psi_{jk}(x)$, with $j \ge 0$, $0 \le k \le 2^j - 1$, form a basis for the space $L^2[0,1]$. Let us denote accordingly $\phi_n(x) = \psi_{jk}(x)$ for $n = 2^j + k$, $0 \le k \le 2^j - 1$, and $\phi_0(x) = 1$ so that $\{\phi_n\}$ form an orthonormal basis for $L^2[0,1]$. Let $Z_n(\omega)$, $n \ge 0$, be a collection of independent Gaussian random variables of mean zero and variance one, that is,

$$P(Z_n < y) = \int_{-\infty}^{y} e^{-y^2} \frac{dy}{\sqrt{2\pi}}.$$

We will show that the process

$$X_t(\omega) = \sum_{n=0}^{\infty} Z_n(\omega) \int_0^t \phi_n(s) ds \tag{1.8}$$

is a Brownian motion.

First, we need to verify that the series (1.8) converges in $L^2(\Omega)$ for a fixed $t \in [0,1]$. Note that

$$\mathbb{E}\left(\sum_{k=n}^{m} Z_k(\omega) \int_0^t \phi_k(s) ds\right)^2 = \sum_{k=n}^{m} \left(\int_0^t \phi_k(s) ds\right)^2 = \sum_{k=n}^{m} \langle \chi_{[0,t]}, \phi_k \rangle^2.$$

As $\phi_k$ form a basis for $L^2[0,1]$, the series (1.8) satisfies the Cauchy criterion and thus converges in $L^2(\Omega)$. Moreover, for any $0 \le s < t \le 1$ we have

$$\begin{aligned}\mathbb{E}\left(X_t - X_s\right)^2 &= \mathbb{E}\left(\sum_{k=0}^{\infty} Z_k(\omega) \int_s^t \phi_k(u) du\right)^2 = \sum_{k=0}^{\infty} \left(\int_s^t \phi_k(u) du\right)^2 = \sum_{k=0}^{\infty} \langle \chi_{[s,t]}, \phi_k \rangle^2 \\ &= \|\chi_{[s,t]}\|_{L^2}^2 = t - s,\end{aligned}$$

hence the increments $X_t - X_s$ have the correct variance. Let us show that they are independent: for $0 \le t_0 < t_1 \le t_2 < t_3 \le 1$:

$$\begin{aligned}\mathbb{E}\left((X_{t_3} - X_{t_2})(X_{t_1} - X_{t_0})\right) &= \mathbb{E}\left(\sum_{k=0}^{\infty} \int_{t_2}^{t_3} \phi_k(u) du \int_{t_0}^{t_1} \phi_k(u') du'\right) \\ &= \sum_{k=0}^{\infty} \langle \chi_{[t_2 t_3]}, \phi_k \rangle \langle \chi_{[t_0 t_1]}, \phi_k \rangle = \langle \chi_{[t_2 t_3]}, \chi_{[t_0 t_1]} \rangle = 0.\end{aligned}$$

As the variables $X_t - X_s$ are jointly Gaussian, independence of the increments follows.

## Continuity of the Brownian motion

In order to prove continuity of the process $X_t(\omega)$ defined by the series (1.8) we show that the series converges uniformly in $t$ almost surely in $\omega$. To this end let us show that

$$M(\omega) = \sup_n \frac{|Z_n(\omega)|}{\sqrt{\log n}} < +\infty \text{ almost surely in } \omega. \tag{1.9}$$

Note that, for each $n \geq 0$:

$$\mathbb{P}\left(|Z_n(\omega)| \geq 2\sqrt{\log n}\right) \leq e^{-(2\sqrt{\log n})^2/2} = \frac{1}{n^2},$$

thus

$$\sum_{n=0}^{\infty} \mathbb{P}\left(|Z_n(\omega)| \geq 2\sqrt{\log n}\right) < +\infty.$$

The Borel-Cantelli lemma implies that almost surely the event $\left\{|Z_n(\omega)| \geq 2\sqrt{\log n}\right\}$ happens only finitely many times, so that $|Z_n(\omega)| < 2\sqrt{\log n}$ for al $n \geq n_0(\omega)$ almost surely, and (1.9) follows.

Another useful observation is that for each fixed $t \geq 0$ and $j \in \mathbb{N}$ there exists only one $k$ so that

$$\int_0^t \phi_{2^j+k}(s)ds \neq 0,$$

and for that $k$ we have

$$\left|\int_0^t \phi_{2^j+k}(s)ds\right| \leq 2^{j/2}2^{-j} = \frac{1}{2^{j/2}}.$$

Hence, we may estimate the dyadic blocs, using (1.9):

$$\left|\sum_{k=0}^{2^j-1} Z_{2^j+k}(\omega)\int_0^t \phi_{2^j+k}(s)ds\right| \leq M(\omega)\sqrt{(j+1)\log 2}\sum_{k=0}^{2^j-1}\left|\int_0^t \psi_{jk}(s)ds\right| \leq \frac{\sqrt{j}M_1(\omega)}{2^{j/2}}.$$

Therefore, the dyadic blocs are bounded by a convergent series which does not depend on $t \in [0,1]$, hence the sum $X_t(\omega)$ of the series is a continuous function for a.e. $\omega$.

### Nowhere differentiability of the Brownian motion

**Theorem 1.1** *The Brownian path $X_t(\omega)$ is nowhere differentiable for almost every $\omega$.*

**Proof.** Let us fix $\beta > 0$. Then if $\dot{X}_s$ exists at some $s \in [0,1]$ and $|\dot{X}_s| < \beta$ then there exists $n_0$ so that

$$|X_t - X_s| \leq 2\beta|t-s| \text{ if } |t-s| \leq \frac{2}{n} \tag{1.10}$$

for all $n > n_0$. Let $A_n$ be the set of functions $x(t) \in C[0,1]$ for which (1.10) holds for some $s \in [0,1]$. Then $A_n \subset A_{n+1}$ and the set $A = \bigcup_{n=1}^{\infty} A_n$ includes all functions $x(t) \in C[0,1]$ such that $|\dot{x}(s)| \leq \beta$ at some point $s \in [0,1]$.

The next step is to replace (1.10) by a discrete set of conditions – this is a standard trick in such situations. Assume that (1.10) holds for a function $x(t) \in C[0,1]$ and let $k = \sup\{j : \ j/n \leq s\}$, then

$$y_k = \max\left(\left|x\left(\frac{k+2}{n}\right) - x\left(\frac{k+1}{n}\right)\right|, \left|x\left(\frac{k+1}{n}\right) - x\left(\frac{k}{n}\right)\right|, \left|x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right|\right) \leq \frac{8\beta}{n}.$$

Therefore, if we denote by $B_n$ the set of all functions $x(t) \in C[0,1]$ for which $y_k \leq 8\beta/n$ for some $k$, then $A_n \subseteq B_n$. Therefore, in order to show that $\mathbb{P}(A) = 0$ it suffices to check that

$$\lim_{n\to\infty} \mathbb{P}(B_n) = 0. \tag{1.11}$$

This, however, can be estimated directly, using translation invariance of the Brownian motion:

$$\begin{aligned}\mathbb{P}(B_n) &\le \sum_{k=1}^{n-2} \mathbb{P}\left[\max\left[\left|X(\frac{k+2}{n}) - X(\frac{k+1}{n})\right|, \left|X(\frac{k+1}{n}) - X(\frac{k}{n})\right|, \left|X(\frac{k}{n}) - X(\frac{k-1}{n})\right|\right] \le \frac{8\beta}{n}\right] \\ &\le n\mathbb{P}\left[\max\left[\left|X\left(\frac{3}{n}\right) - X\left(\frac{2}{n}\right)\right|, \left|X\left(\frac{2}{n}\right) - X\left(\frac{1}{n}\right)\right|, \left|X\left(\frac{1}{n}\right)\right|\right] \le \frac{8\beta}{n}\right] \\ &= n\mathbb{P}\left[\left|X\left(\frac{1}{n}\right)\right| \le \frac{8\beta}{n}\right]^3 = n\left(\sqrt{\frac{n}{2\pi}} \int_{-8\beta/n}^{8\beta/n} e^{-nx^2/2} dx\right)^3 \le n\left(\sqrt{\frac{n}{2\pi}} \frac{16\beta}{n}\right)^3 \le \frac{C}{\sqrt{n}},\end{aligned}$$

which implies (1.11). It follows that $\mathbb{P}(A) = 0$ as well, hence Brownian motion is nowhere differentiable with probability one. □

**Corollary 1.2** *Brownian motion does not have bounded variation with probability one.*