Problem 1. (12 pts.) Given a parametrized path \( \vec{\alpha}(t) = (2t, 1 - t, t^2) \) for \( 0 \leq t \leq 1 \) compute

a) \( \int_{\vec{\alpha}} x \, ds \)

**Solution:** Let \( X(t) = (2t, 1 - t, t^2) \), then \( X'(t) = (2, -1, 2t) \), and \( ||X'(t)|| = \sqrt{4 + 1 + 4t^2} = \sqrt{5 + 4t^2} \).

The integral is thus:

\[
I = \int_{0}^{1} 2t\sqrt{5 + 4t^2} \, dt = \int_{0}^{1} \sqrt{5 + 4u} \, du \quad (u = t^2)
\]

\[
I = \frac{1}{6} \left( (5 + 4u)^{\frac{3}{2}} \right)_{0}^{1} = \frac{27 - 5\sqrt{5}}{6}
\]

b) \( \int_{\vec{\alpha}} x \, dx \)

**Solution:** Here the differential element \( dx \) is more simple:

\[
I = \int_{0}^{1} 2t \, 2 \, dt = 2
\]

Problem 2. (10 pts.) Let \( \vec{F} = (3x + 8y) \, \hat{i} + (5x - y) \, \hat{j} \).

Evaluate \( \int_{C} \vec{F} \cdot \vec{n} \, ds \) for \( C \) being the circle \( x^2 + y^2 = 4 \) and \( \vec{n} \) the normal vector pointing out.

**Hint:** use the divergence version of Green’s Theorem.
Solution: Here we will use the divergence theorem. According to the divergence theorem, this integral is equal to the double integral of $\nabla F$ over the disk $D$ of radius 2. Here $\nabla F = 3 - 1 = 2$. So the integral is:

$$I = \iint_D 2 \, dx \, dy$$

that is the integral is twice the area of the disk:

$$I = 2\pi 2^2 = 8\pi$$

Problem 3. (10 pts.) Let $\vec{F}$ be the vector field $\vec{F} = \frac{-y + x}{x^2 + y^2} \, \hat{i} + \frac{x + y}{x^2 + y^2} \, \hat{j}$.

a) Evaluate $\oint_C \vec{F} \cdot d\vec{s}$ for $C$ being the circle $x^2 + y^2 = 4$ oriented counter clockwise.

Solution: Parametrize $C$ by $\alpha(t) = (2 \cos t, 2 \sin t)$ with $0 \leq t \leq 2\pi$. Since $t$ will represent the angle between positive x-axis, $\alpha(t)$ moves along $C$ as $t$ varies from 0 to $2\pi$. Then,

$$\int_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\alpha(t)) \cdot \alpha'(t) \, dt = \int_0^{2\pi} \vec{F}(2 \cos t, 2 \sin t) \cdot (-2 \sin t, 2 \cos t) \, dt$$

$$= \int_0^{2\pi} \frac{-2 \sin t + 2 \cos t}{4} \cdot (-2 \sin t) + \frac{2 \cos t + 2 \sin t}{4} \cdot 2 \cos t \, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi$$

Remark. Green’s theorem (or $\nabla \times \vec{F} = 0$) will not apply in this case, because $\vec{F}$ is not defined on the region where $C$ encloses, namely on $\{ x^2 + y^2 \leq 4 \}$.

b) Is $\vec{F}$ conservative on $\mathbb{R}^2 \setminus \{ 0 \}$? Explain.

Solution: No. If $\vec{F}$ were conservative, integration of $\vec{F}$ along any closed curve would vanish by the Fundamental Theorem of Line Integral. But in 3.a, we have shown there is a closed curve on which $\vec{F}$ does not vanish, thus $\vec{F}$ cannot be conservative.

Remark: $\nabla \times F = 0$ is not sufficient to prove conservativity in this case, since $\mathbb{R}^2 \setminus \{ (0,0) \}$ is not simply connected.

Remark2: However, the fact that $\mathbb{R}^2 \setminus \{ (0,0) \}$ is not simply connected is not sufficient in proving $\vec{F}$ is not conservative. This simply means that $\vec{F}$ need not be conservative. Note that there are many conservative defined on non-simply connected region. For example, $\vec{F} = \left( \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right)$ is conservative, because for $f = \ln(x^2 + y^2)$, you can check that $\vec{F} = \nabla f$, and of course, integration along any closed curve lie in $\mathbb{R}^2 \setminus \{ (0,0) \}$ will vanish whether it encloses the origin or not, by fundamental theorem of line integral.)
Problem 4. (10 pts.) Let, like before, \( \vec{F} = \frac{-y + x}{x^2 + y^2} \hat{i} + \frac{x + y}{x^2 + y^2} \hat{j} \).

a) Show that \( \vec{F} \) is conservative on \( \{ (x, y) \mid x > 0 \} \).

Solution: First of all, we can see \( \{ (x, y) : x > 0 \} \) is simply connected, as it has no holes in it. Therefore, \( \vec{F} \) is conservative if and only if \( \nabla \times \vec{F} = 0 \), or \( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial x} \) where \( \vec{F} = (M, N) \).

Let \( \vec{F} = Mi + Nj \). Then,

\[
\frac{\partial N}{\partial x} - \frac{\partial M}{\partial x} = \frac{(x^2 + y^2) - 2x(x + y)}{(x^2 + y^2)^2} - \frac{-(x^2 + y^2) - 2y(-y + x)}{(x^2 + y^2)^2} \\
= \frac{(-x^2 + y^2 - 2xy) - (-x^2 + y^2 - 2xy)}{(x^2 + y^2)^2} = 0
\]

Therefore \( \vec{F} \) is conservative.

Alternate solution: Let \( f(x, y) = \arctan\left(\frac{y}{x}\right) + \frac{1}{2} \ln(x^2 + y^2) \) defined on \( x > 0 \). Then by direct computation,

\[
\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} + \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} = \frac{-y + x}{x^2 + y^2} \\
\frac{\partial f}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2y}{x^2 + y^2} = \frac{x + y}{x^2 + y^2}
\]

Therefore \( \vec{F} = \nabla f \) is conservative.

b) Evaluate \( \oint_C \vec{F} \cdot d\vec{s} \) for \( C \) being the circle \( (x-3)^2 + (y+3)^2 = 4 \) oriented counter clockwise.

Solution:
The closed curve \( C \) lies entirely in the domain \( x > 0 \) on which \( \vec{F} \) is conservative. Therefore, by fundamental theorem of line integral, \( \int_C \vec{F} \cdot ds = 0 \)

Alternate solution: \( \vec{F} \) is defined on the curve \( C \) as well as the interior of the domain enclosed by \( C \). Therefore, Green's theorem will apply. Thus,

\[
\int_C \vec{F} \cdot ds = \int \int_{(x-3)^2 + (y+3)^2 \leq 4} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial x} \right) dxdy = 0
\]

Alternate solution 2 (Just for fun): Parametrize \( C \) by \( \alpha(t) = (3 + 2 \cos t, -3 + 2 \sin t) \) with \( 0 \leq t \leq 2\pi \). Then,

\[
I := \int_C \vec{F} \cdot ds = \int_0^{2\pi} \frac{4 - 12 \sin t}{22 + 12(\cos t - \sin t)} dt
\]
The integrand is periodic with period $2\pi$, then it is not harmful to translate $t$ by $-\pi$. Thus,

$$I = \int_{-\pi}^{\pi} \frac{4 - 12 \sin t}{22 + 12(\cos t - \sin t)} dt$$

Now, use trigometric substitution $u = \tan(t/2)$. This is general technique to convert trigonometric integral into integration of rational functions. Then we have the following rules:

$$\sin t = \frac{2u}{1 + u^2} \quad \cos t = \frac{1 - u^2}{1 + u^2} \quad du = \frac{1}{2} \sec^2(t/2) = \frac{1}{2}(1 + u^2) dt,$$

and the limit is from $-\infty$ to $\infty$

$$I = \int_{-\infty}^{\infty} \frac{4 - 12 \frac{2u}{1 + u^2}}{22 + 12 \frac{1 - u^2}{1 + u^2} - 12 \frac{1}{1 + u^2}} \cdot \frac{2}{1 + u^2} du$$

$$= \int_{-\infty}^{\infty} \frac{8 - 48u + 8u^2}{(17 - 12u + 5u^2)(1 + u^2)} du$$

$$= \int_{-\infty}^{\infty} \frac{10u - 26}{5u^2 - 12u + 17} - \frac{2u - 2}{1 + u^2} du$$

$$= \int_{-\infty}^{\infty} \frac{10u - 12}{5u^2 - 12u + 17} - \frac{2u}{1 + u^2} du - 14 \int_{-\infty}^{\infty} \frac{1}{5u^2 - 12u + 17} du + 2 \int_{-\infty}^{\infty} \frac{1}{1 + u^2} du$$

$$= II + III + IV$$

$$II = \ln \left. \frac{5u^2 - 12u + 17}{u^2 + 1} \right|_{-\infty}^{\infty} = \ln 5 - \ln 5 = 0$$

$$III = -\frac{14}{5} \int_{-\infty}^{\infty} \frac{1}{(u - \frac{6}{5})^2 + \left( \frac{7}{5} \right)^2} du = -\frac{14}{5} \cdot \frac{5}{7} \arctan \left. \frac{7}{5} (u - \frac{6}{5}) \right|_{-\infty}^{\infty} = -2\pi$$

$$IV = 2 \arctan u \bigg|_{-\infty}^{\infty} = 2\pi$$

Therefore $I = II + III + IV = 0$.

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**Problem 5.** (10 pts.) Let, like before, $\vec{F} = \frac{-y + x}{x^2 + y^2} \vec{i} + \frac{x + y}{x^2 + y^2} \vec{j}$.

Evaluate $\int_C \vec{F} \circ \vec{n} \ ds$ for $C$ being the circle $x^2 + y^2 = 4$ and $\vec{n}$ the normal vector pointing out.
Solution: The Green’s Theorem does not apply for the curve $C$ as $\vec{F}$ is not defined at the origin. We integrate along a circle centered at the origin, so $\vec{n} = \frac{1}{2} \left( \vec{x} + \vec{y} \right)$. 

On the circle $C$, $x^2 + y^2 = 4$, so $\vec{F} = \frac{-y + x}{4} \vec{i} + \frac{x + y}{4} \vec{j}$ and hence 

$$\vec{F} \cdot \vec{n} = \frac{1}{8} \left( -xy + x^2 + xy + y^2 \right) = \frac{x^2 + y^2}{8} = \frac{1}{2}$$

Thus 

$$\int_C \vec{F} \cdot \vec{n} \, ds = \frac{1}{2} \cdot \text{length of circle radius 2} = \frac{1}{2} 2\pi \cdot 2$$

One could also parametrize the curve $C$ with $(2 \cos t, 2 \sin t)$ with $\vec{n} = \cos t \vec{i} + \sin t \vec{j}$. Then $ds = 2 \, dt$ and:

$$\int_C \vec{F} \cdot \vec{n} \, ds = \int_0^{2\pi} \frac{-2 \sin t + 2 \cos t}{4} \cdot \cos t + \frac{2 \cos t + 2 \sin t}{4} \cdot \sin t \, 2dt$$

Problem 6. (12 pts.) Let $C$ be the broken line going first from $(0, 0)$ to $(0, b)$ then to $(a, b)$ and ending at $(a, 0)$, where $a$ and $b$ are some positive numbers.

![Diagram of a broken line representing $C$.]

Show that for any smooth function $f(x)$:

$$\int_C (y \cdot f'(x)) \, dx + (x + f(x)) \, dy = -ab$$

Solution: The easiest is to complete $C$ to a closed path by adding a line from $(a, 0)$ to $(0, 0)$. Let’s denote this line by $L$. Then we can parametrize $-L$ by $t \to (t, 0)$ for $0 \leq t \leq a$ so

$$\int_{-L} (y \cdot f'(x)) \, dx + (x + f(x)) \, dy = \int_0^a 0 \cdot f'(t) \, dt + (t + f(t)) \, 0 \cdot dt = 0$$

so

$$\int_{C \cup L} (y \cdot f'(x)) \, dx + (x + f(x)) \, dy = \int_C (y \cdot f'(x)) \, dx + (x + f(x)) \, dy$$

Now we can apply Green’s Theorem to the closed path $-C \cup -L$ bounding the rectangle $R$:

$$\int_{-C \cup -L} (y \cdot f'(x)) \, dx + (x + f(x)) \, dy = \iint_R -f'(x) + 1 + f'(x) \, dA = ab$$
from which the result follows.
“Applying” Green’s theorem to the curve $C$ (which is not closed) yielded only half credit, which is still generous.

One can also parametrize each piece of $C$ directly to get

$$
\int_C (y \cdot f'(x)) \, dx + (x + f(x)) \, dy = f(0) \cdot b + b \cdot (f(a) - f(0)) - b \cdot (a + f(a)) = -ab
$$

For some strange reason many students assumed $f(0) = 0$, for which credit was taken off.

**Problem 7.** (12 pts.) Let $\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$ (the position vector), and let $r$ be its length.

a) Show that there is no vector field $\vec{F}$ such that

$$
\nabla \times \vec{F} = \vec{r}
$$

**Solution:** We prove this by contradiction. Suppose there exists a vector field $\vec{F}$ such that $\nabla \times \vec{F} = \vec{r}$. Since $\nabla \cdot (\nabla \times \vec{F}) = 0$ for any twice differentiable vector field $\vec{F}$, we have

$$
0 = \nabla \cdot (\nabla \times \vec{F}) = \nabla \cdot \vec{r} = 3
$$

which is a contradiction.

b) Find the number $\alpha$ such that

$$
\nabla \circ \left( \frac{\vec{r}}{r^\alpha} \right) = 0
$$

**Solution:** By product rule and chain rule,

$$
\nabla \cdot \left( \frac{\vec{r}}{r^\alpha} \right) = \nabla \left( \frac{1}{r^\alpha} \right) \cdot \vec{r} + \frac{1}{r^\alpha} (\nabla \cdot \vec{r})
$$

$$
= \left( \frac{-\alpha}{r^{\alpha+1}} \nabla r \right) \cdot \vec{r} + \frac{3}{r^\alpha}
$$

$$
= \left( \frac{-\alpha}{r^{\alpha+1}} \frac{\vec{r}}{r} \right) \cdot \vec{r} + \frac{3}{r^\alpha}
$$

$$
= \frac{-\alpha + 3}{r^\alpha}
$$

Therefore,

$$
\nabla \cdot \left( \frac{\vec{r}}{r^\alpha} \right) = 0 \quad \text{if and only if} \quad \alpha = 3.
$$
Problem 8. (12 pts.) Let $\Gamma$ be a curve parametrized by 

$$\gamma(t) = (t, 0, f(t))$$

for some smooth function $f(t)$ and $0 < a \leq t \leq b$.

(a) Write an integral of the form $\int \ldots \, dt$ representing the length of the curve $\Gamma$.

**Solution:** In general, the length of a parameterized curve $(x(t), y(t), z(t))$ is given by 

$$\int \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.$$ 

So, in this case, the length is 

$$\int_a^b \sqrt{1 + f'(t)^2} \, dt.$$ 

(b) Use additional angle parameter $\theta$ to parametrize the surface $S$ obtained by rotating the curve $\Gamma$ about the $z$ axis.

**Solution:** There are many ways to parameterize this curve, but one that makes the axes 'look the same' seems like a good plan. One way to do that is to have an angle $\theta$, and for each fixed $\theta$ draw the curve through the plane containing the $z$-axis and at angle $\theta$ to the $x$-axis. This results in the parameterization $(t \cos(\theta), t \sin(\theta), f(t))$, $a \leq t \leq b$, $0 \leq \theta \leq 2\pi$.

(c) Use the results of parts a) and b) to show that the surface area of $S$ is equal to 

$$2\pi L \bar{x}$$

where $L$ is the length of the curve $\Gamma$ and $\bar{x}$ is the $x$ coordinate of the centroid of $\Gamma$.

**Solution:** Using the above parameterization, we find tangent vectors $T_t = (\cos(\theta), \sin(\theta), f'(t))$ and $T_\theta = (-t \sin(\theta), t \cos(\theta), 0)$ and so normal vector $N_{t\theta} = (-tf'(t) \cos(\theta), -tf'(t) \sin(\theta), t)$. Simplifying a bit, and noting that $t \geq 0$, the normal vector has norm $||N_{t\theta}|| = t \sqrt{f'(t)^2 + 1}$. Thus, the surface area is given by 

$$A = \int_0^{2\pi} \int_a^b t \sqrt{f'(t)^2 + 1} \, dt \, d\theta$$

$$= \int_0^{2\pi} \left[ \int_a^b \sqrt{f'(t)^2 + 1} \, dt \right] t \, d\theta$$

$$= 2\pi L \bar{x}$$

where the second line comes from multiplying and dividing by the 'total mass' $L = \int_a^b \sqrt{f'(t)^2 + 1} \, dt$.

Problem 9. (12 pts.) Let $S$ be the part of the cone $z = 3 - 2\sqrt{x^2 + y^2}$ that is above the plane $z = 0$. 

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[7]
a) Find a parameterization of $S$, for which the normal vector points away from the $z$ axis.

**Solution:** Notice that the surface $S$ is the graph of the function $f(x,y) = 3 - 2\sqrt{x^2 + y^2}$ over the domain $D = \{(x,y) : x^2 + y^2 \leq \frac{9}{4}\}$. Define a parametrization by

$$\vec{X}(s,t) = (s,t,3-2\sqrt{s^2+t^2})$$

where $(s,t) \in D$. Then,

$$\vec{T}_s = \left(1, 0, \frac{-2s}{\sqrt{s^2+t^2}}\right)$$

$$\vec{T}_t = \left(0, 1, \frac{-2t}{\sqrt{s^2+t^2}}\right)$$

$$\vec{N}_{s,t} = \vec{T}_s \times \vec{T}_t = \left(\frac{2s}{\sqrt{s^2+t^2}}, \frac{2t}{\sqrt{s^2+t^2}}, 1\right)$$

Note that $N_{s,t} \cdot (0,0,1) = 1 > 0$. So the normal vector points out of the cone, hence away from the $z$-axis.

OR

One could use polar coordinates:

$$\vec{X}(r, \theta) = (r \cos \theta, r \sin \theta, 3 - 2r)$$

where $0 \leq r \leq \frac{3}{2}$ and $0 \leq \theta \leq 2\pi$. Then,

$$\vec{T}_r = (\cos \theta, \sin \theta, -2)$$

$$\vec{T}_\theta = (-r \sin \theta, r \cos \theta, 0)$$

$$\vec{N}_{r,\theta} = \vec{T}_r \times \vec{T}_\theta = (2r \cos \theta, 2r \sin \theta, r)$$

Since $N_{r,\theta} \cdot (0,0,1) = r > 0$, the normal vector points out of the cone, hence away from the $z$-axis.

b) Compute $\iint_S F \cdot dS$, where $S$ is oriented such that the normal vector points away from $z$ axis, and $F(x,y,z) = x \cdot \vec{i} + y \cdot \vec{j} + z \cdot \vec{k}$.

You can “recycle” the results from the Problem 8 (b-c).

**Solution:**

$$\vec{F}(\vec{X}(s,t)) = (s,t,3-2\sqrt{s^2+t^2})$$

Therefore,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{X}(s,t)) \cdot \vec{N}_{s,t} \ dA$$

$$= \iint_D 3 \ dA$$

$$= 3 \ \text{Area}(D) = \frac{27\pi}{4}.$$