SOME SAMPLE EXAM PROBLEMS

1. Let $A$ be an $n \times n$ self-adjoint matrix. (That is, assume $A^* = A$.) Prove that all the eigenvalues of $A$ are real.

2. Suppose $A$ is a self-adjoint $n \times n$ matrix. Suppose $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors with different eigenvalues. Prove that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

3. Prove (as in 2) that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, assuming $A$ is normal (i.e., $AA^* = A^*A$) but not necessarily self-adjoint.

4. Let $V$ be a finite-dimensional complex vector space and $T : V \rightarrow V$ be a linear operator. Suppose the only eigenvalue of $T$ is 0. Prove that $T^k = 0$ for some $k$.

5. Let $A$ be an $n \times n$ real matrix such that $Ax$ is nonzero and perpendicular to $x$ for every nonzero $x$. Prove that $n$ is even.

6. Suppose for a certain matrix $A$ that

$$A^2 - 3A + 2I = 0.$$ 

(a) Prove that if $\lambda$ is an eigenvalue of $A$, then $\lambda = 1$ or $\lambda = 2$.
(b)* Prove that either 1 and 2 must be an eigenvalue of $A$. (Both may be.)

7. Suppose $T : V \rightarrow V$ is a linear operator on $d$-dimensional complex vector space $V$ and that $T$ has $d$ distinct eigenvalues $\lambda_1, \ldots, \lambda_d$. Prove that

$$T^n \mathbf{v} \rightarrow 0 \quad \text{for all } \mathbf{v}$$

if and only if $|\lambda_i| < 1$ for every $i$.

8. Suppose $V$ is a finite dimensional vector space and that $W$ is a $k$-dimensional subspace. Prove that there is a basis for $V$ whose first $k$ elements are a basis for $W$.

9. Let $M$ be a $3 \times 3$ real symmetric matrix with eigenvectors $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (-1, -1, 1)$. Find a third eigenvector (not just a scalar multiple of $\mathbf{u}$ or $\mathbf{v}$.)

10. Let $T$ be a $3 \times 3$ matrix all of whose entries are positive real numbers. Prove that $T$ has an eigenvector whose components are all nonnegative real numbers. (In fact the components will all be positive.)

11. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a bounded continuous map. (Bounded means there is an $M < \infty$ such that $|F(x)| \leq M$ for all $x$.) Prove there is an $x$ such that $F(x) = x$. 

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