**Definition.** A non-empty subset $V$ of $\mathbb{R}^n$ is called a **linear subspace** if and only if it is closed under addition and under scalar multiplication, i.e., if and only if

$$
A, B \in V \implies A + B \in V \\
A \in V \text{ and } c \in \mathbb{R} \implies cA \in V
$$

**Remark.** If $V$ is a subspace, then any linear combination of vectors in $V$ must also be in $V$. For suppose $A_1, \ldots, A_k$ are vectors in $V$ and $c_1, \ldots, c_k$ are scalars. Then $c_iA_i$ is in $V$ (closure under scalar multiplication) for each $i$. Therefore $\sum c_iA_i$ is also in $V$ (by closure under addition).

**A Convention**

Recall that if $S$ is a set of vectors in $\mathbb{R}^n$, then $L(S)$ is the set of all linear combinations of vectors in $S$. What if $S$ is the empty set $\emptyset$? One might well think that $L(\emptyset)$ should be the empty set, but it turns out to be much better to define $L(\emptyset)$ to be the set $\{0\}$ containing just the origin.

As someone observed in class, this convention is very natural if we think of $L(S)$ as all the points we can reach by starting at the origin and moving backwards and forwards in the directions given by the vectors in $S$.

**Theorem 1.** If $S = \{A_1, \ldots, A_k\}$ is a set of vectors in $\mathbb{R}^n$, then $L(S)$ is a subspace of $\mathbb{R}^n$.

**Proof.** First note that $L(S)$ is not empty since it must contain the origin.

Now suppose $X$ and $Y$ are in $L(S)$. Then $X = \sum c_iA_i$ and $Y = \sum d_iA_i$ for suitable scalars $c_i$ and $d_i$. Thus

$$
X + Y = \sum c_iA_i + \sum d_iA_i = \sum (c_i + d_i)A_i
$$

so $X + Y \in L(S)$. This proves closure under addition. Likewise if $r$ is a scalar, then

$$
rX = r\sum c_iA_i = \sum (rc_i)A_i
$$

so $rX$ is in $L(S)$. This proves closure under scalar multiplication. □

**Lemma.** Let $S$ be a list of vectors in $\mathbb{R}^n$. If one of the vectors, say $X$, in $S$ is a linear combination of the others. Then removing it from $S$ gives a new set $S'$ with the same linear span:

$$L(S) = L(S')$$

**Proof.** Let $A_1, \ldots, A_k, X$ be the vectors in $S$, so $S' = \{A_1, \ldots, A_k\}$. Since $X$ is a linear combination of the $A_i$'s,

$$X = \sum c_iA_i$$
for suitable scalars $c_i$.
Now suppose $Y \in L(S)$. Then

$$Y = \sum d_i A_i + rX$$

for suitable scalars $d_i$ and $r$. Thus

$$Y = \sum d_i A_i + r(\sum c_i A_i) = \sum (d_i + rc_i) A_i$$

so $Y \in L(S')$. This shows that every vector in $L(S)$ is also in $L(S')$.

However, it is clear (why?) that every vector in $L(S')$ is also in $L(S)$. Thus $L(S')$ and $L(S)$ are the same. □

**Theorem 2.** Suppose $B_1, B_2, \ldots, B_\ell$ are linearly independent vectors in $L(A_1, \ldots, A_k)$. Then $\ell \leq k$.

**Proof.** Since $B_1$ is in $L(A_1, \ldots, A_k)$, the vectors

$$B_1, A_1, \ldots, A_k$$

are dependent. Thus one of them is a linear combination of the preceding ones. Note that one cannot be $B_1$. (Otherwise $B_1$ would be 0, which is impossible since the $B$’s are independent.) Thus it must be one of the $A$’s. By relabelling the $A$’s, if necessary, we may assume it’s $A_k$.

Now by the lemma,

$$L(A_1, \ldots, A_k) = L(B_1, A_1, \ldots, A_k) = L(B_1, A_1, \ldots, A_{k-1})$$

Thus

$$B_2, \ldots, B_{\ell-1} \in L(B_1, A_1, \ldots, A_{k-1})$$

By the same argument, the vectors

$$B_2, B_1, A_1, \ldots, A_{k-1}$$

are dependent, so one is a linear combination of the preceding ones, that one cannot be one of the $B$’s, so it must be an $A_i$. By relabelling, we may assume it is $A_{k-1}$. Then by the lemma,

$$L(B_1, A_1, \ldots, A_{k-1}) = L(B_2, B_1, A_1, \ldots, A_{k-2}) = L(B_2, B_1, A_1, \ldots, A_{k-2})$$

Consequently,

$$B_3, \ldots, B_\ell \in L(B_2, B_1, A_1, \ldots, A_{k-2})$$

Now repeat the process. If $k$ were less than $\ell$, then after $k$ steps we would have

$$B_{k+1}, \ldots, B_\ell \in L(B_k, B_{k-1}, \ldots, B_1)$$

which is impossible (since the $B$’s are independent).

Thus $k \geq \ell$. □

**Corollary 3.** If $B_1, \ldots, B_\ell$ are linearly independent vectors in $\mathbb{R}^n$, then $\ell \leq n$.

**Proof.** Since $B_1, \ldots, B_\ell$ are in $L(E_1, \ldots, E_n)$, $\ell \leq n$ by theorem 3. □
Basis of a Subspace

**Definition.** A *basis* for a subspace $V$ is a linearly independent set $S$ of vectors whose span is $V$:

$$L(S) = V$$

**Theorem 3.** Suppose $V$ is a subspace of $\mathbb{R}^n$. Then

1. There is a basis for $V$.
2. Any two bases for $V$ have the same number of elements.

*Proof of (1).* If $V = \{0\}$, we let $S$ be the empty set.

Now suppose $V$ has nonzero elements. We define a sequence $A_1, A_2, \ldots$ of vectors in $V$ as follows.

Let $A_1$ be any nonzero vector in $V$.

Given $A_1, \ldots, A_k$, if $L(A_1, \ldots, A_k) = V$, we stop. If not, then there is a vector that belongs to $V$ but not to $L(A_1, \ldots, A_k)$. Let $A_{k+1}$ be such a vector.

Note that the set of vectors we get is linearly independent, since no one them is a linear combination of the preceding ones. The procedure must stop after at most $n$ steps (since any set of more than $n$ vectors in $\mathbb{R}^n$ must be dependent by corollary 3.)

*Proof of (2).* Let $B_1, \ldots, B_\ell$ be one basis for $V$ and $A_1, \ldots, A_k$ be another basis for $V$. Then

$$B_1, \ldots, B_\ell \in L(A_1, \ldots, A_k)$$

so by theorem 2, $\ell \leq k$. Likewise $k \leq \ell$. □

**Definition.** Suppose $V$ is a subspace of $\mathbb{R}^n$. The *dimension* of $V$, written $\dim V$, is the number of vectors in a basis for $V$.

According to theorem 3, every subspace has one and only one dimension.

**Examples**

The empty set $\emptyset$ is a basis for $\{0\}$, so $\dim \{0\} = 0$.

The vectors $E_1, \ldots, E_n$ form a basis for $\mathbb{R}^n$, so $\dim \mathbb{R}^n = n$. 