Consider a $C^2$ curve $X : [a, b] \to \mathbb{R}^N$ with $X'$ never 0. Recall that the curvature vector $K(t)$ is defined to be the derivative of the unit tangent vector with respect to arclength:

$$K(t) = \frac{dT}{ds},$$

where

$$T = \frac{X'}{|X'|}.$$  

This definition looks simple, but using it to calculate curvature is usually messy. The formula for arclength is itself messy since it is an integral of a square root. The expression for $T$ is also complicated, since it is a fraction with a square root in the denominator.

The following two theorems give a method for calculating curvature that is usually easier.

**Theorem 1.** Consider a curve $X : [a, b] \to \mathbb{R}^N$ such that $X'$ and $X''$ exist and such that $X'$ is never 0. Then

$$X'' = \frac{dv}{dt} X' + v^2 K$$

where $v(t) = |X'(t)|$ is the speed at time $t$.

**Proof.** Since $v = |X'|$,

$$T = \frac{X}{|X'|} = \frac{X}{v},$$

so

$$X' = vT.$$  

Differentiate this last equation with respect to $t$:

$$X'' = \frac{dv}{dt} T + v \frac{dT}{dt}$$

$$= \frac{dv}{dt} T + v \frac{dT}{ds} \frac{ds}{dt}$$

$$= \frac{dv}{dt} T + v^2 K$$

since $ds/dt = v$.  □
Theorem 2. Let $X : [a, b] \to \mathbb{R}^N$ be a curve as in theorem 1. Then

$$K = \frac{X''}{v^2} - \frac{X'' \cdot X'}{v^4} X'. $$

Proof. By theorem 1,

\[(*) \quad X'' = \frac{dv}{dt} T + v^2 K. \]

Recall that $K$ is orthogonal to $T$. Thus if we multiply $(*)$ by $T$ we get:

$$X'' \cdot T = \frac{dv}{dt} T \cdot T + 0 = \frac{dv}{dt}. $$

Plugging this expression for $dv/dt$ into $(*)$ gives

$$X'' = (X'' \cdot T) T + v^2 K$$

or

$$v^2 K = X'' - (X'' \cdot T) T$$

$$= X'' - \left(\frac{X'' \cdot \frac{X'}{v}}{v}\right) \frac{X'}{v}$$

Now divide both sides by $v^2$. \[\square\]

Example. Find the curvature of the curve $X(t) = (\cos t, \sin t, e^t)$ when $t = 0$.

Solution. Differentiation gives

$$X'(t) = (-\sin t, \cos t, e^t)$$

$$X''(t) = (-\cos t, -\sin t, e^t)$$

$$X'(0) = (-\sin 0, \cos 0, e^0) = (0, 1, 1)$$

$$v(0) = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$$

$$X''(0) = (-\cos 0, -\sin 0, e^0) = (-1, 0, 1).$$

Hence by theorem 2 at time $t = 0$,

$$K = \frac{X''^2}{v^2} - \frac{X'' \cdot X'}{v^4} X'$$

$$= \frac{(-1, 0, 1)}{2} - \frac{(-1, 0, 1) \cdot (0, 1, 1)}{4}(0, 1, 1)$$

$$= \left(-\frac{1}{2}, 0, \frac{1}{2}\right) - \frac{1}{4}(0, 1, 1)$$

$$= \left(-\frac{1}{2}, 0, -\frac{1}{4}\right).$$

(To check our arithmetic so far, we can take the dot product of this vector with $X'(0)$. The result is 0, which it should be since the curvature vector is always orthogonal to the velocity.)

Of course the scalar curvature is

$$\kappa = |K| = |\left(-\frac{1}{2}, 0, -\frac{1}{4}\right)| = \frac{1}{4} |(-2, 1, 1)| = \frac{1}{4} \sqrt{4 + 1 + 1} = \frac{\sqrt{6}}{4}. $$