Complete the following definitions:

1(a). An **inductive set** is a set $X$ such that: $0 \in X$ and $y \in X \rightarrow S(y) \in X$ for all $y$.

1(b). The ordered pair $(x, y) = \{x, \{x, y\}\}$. *Notes:* A large number of people gave the definition as $(x, y) = \{x, \{x, y\}\}$. This is not the correct definition, but it is an “good” definition in the sense that $(a, b) = (c, d)$ implies $a = c$ and $b = d$ if one assumes Foundation. It received two points of five possible.

2(a). A **relation** from $A$ to $B$ is: a set of ordered pairs $(x, y)$ such that $x \in A$ and $y \in B$. Also acceptable: a subset of $A \times B$.

2(b). A **natural number** is a set $x$ such that: $x \in I$ for every inductive set $I$. Also accepted: $x \in \mathbb{N}$ where $\mathbb{N}$ is the intersection of all inductive sets.

3(a). Let $x$ be any set. The **successor** of $x$ is: $S(x) = x \cup \{x\}$.

3(b). A **transitive set** is a set $S$ such that: for all $x, y$ where $x \in y \in S$, we have $x \in S$. Also accepted: for all $y \in S$, $y \subseteq S$.

4. Prove carefully that there is no set $S$ such that $x \in S$. Be sure you mention each axiom that you use.

*Solution:* Suppose such a set $S$ exists. By a Selection axiom, there is a set $A = \{x \in S : x \notin x\}$. Since $A \in S$, we have $A \in A$ iff $A \notin A$, a contradiction. Therefore no such $S$ exists. □

*Notes:* Several people attempted to reach a contradiction through the Axiom of Foundation, which was acceptable provided that it was done correctly (which I think happened exactly once). The most common error was to apply Foundation to $S$, which is fruitless because $S$ actually does contain an $\epsilon$-minimal element, namely $\emptyset$. Instead, one can apply the axiom to $T = \{S\}$, which is a counterexample to Foundation.

5. Recall that the addition of natural numbers is defined so that (i) $n + 0 = n$ and (ii) $n + S(m) = S(n + m)$. Prove from this definition that $0 + n = n$ for every natural number $n$. [You should prove this “from scratch”, i.e., without using facts about addition, subtraction, etc.]

*Solution:* Prove by induction on $n$. For the base case $n = 0$ we have (by rule (i) with $n = 0$) $0 + 0 = 0$, as desired. For the inductive case, assume the theorem holds for $n = k$ and prove it for $n = S(k)$. We have $0 + S(k) = S(0 + k)$ by rule (ii) and $S(0 + k) = S(k)$ by IH (that $0 + k = k$), so $0 + S(k) = S(k)$ as desired. □
6. Show that $N \times N \lesssim N$ (i.e., $|N \times N| \leq |N|$) by giving an example of a function $f: N \times N \to N$ that is one-to-one.

**Solution:** Any injection $f$ was acceptable, as long as some argument was given to show that $f$ was actually 1-1. The most common example was $f(m, n) = 2^m3^n$, but I also saw a couple of instances of $f(m, n) = (m + n)(m + n + 1)/2 + n$ and one instance of $f(m, n) = 2^{m(2n + 1)} - 1$. Variants of these that were not injections (e.g., $f(m, n) = m^2n^3$), were given low scores, since to me they demonstrated memorization without comprehension.

7. Prove that $2^N$ and $\mathcal{P}(N)$ have the same cardinality by giving a bijection from one to the other.

**Solution:** Let $f: 2^N \to \mathcal{P}(N)$ be defined by $f(\varphi) = \varphi^{-1}[\{1\}] = \{n \in N : \varphi(n) = 1\}$. Let $\chi: \mathcal{P}(N) \to 2^N$ be defined by

$$\chi_S(n) = \begin{cases} 1, & n \in S \\ 0, & n \notin S \end{cases}$$

We must show that $f$ and $\chi$ are inverses, i.e., that $f \circ \chi = 1_{2^N}$ and $\chi \circ f = 1_{\mathcal{P}(N)}$. But

$$f(\chi_S) = \{n \in N : \chi_S(n) = 1\} = S$$

and since, for all $n \in N$,

$$\chi_{f(\varphi)}(n) = \begin{cases} 1, & n \in f(\varphi) \\ 0, & n \notin f(\varphi) \end{cases} = \begin{cases} 1, & \varphi(n) = 1 \\ 0, & \varphi(n) = 0 \end{cases} = \varphi(n),$$

it follows that $\chi_{f(\varphi)} = \varphi$, as desired. □

**Notes:** The above argument, of course, has nothing to do with $N$, and would work as well with any set $A$. The most common error on this problem was to attempt to find bijections from $2^N$ to $\mathbb{R}$ and from $\mathbb{R}$ to $\mathcal{P}(N)$, which is practically impossible.

8. Prove that $\mathcal{P}(N)$ and $\mathcal{P}(N)^N$ have the same cardinality. [Don’t just say “because $\mathbb{R}$ and $\mathbb{R}^N$ have the same cardinality”.

**Solution:**

$$\mathcal{P}(N)^N \sim (2^N)^N \sim 2^{N \times N} \sim 2^N \sim \mathcal{P}(N).$$

The first equality is by problem 7 and a standard identity, the second by another identity, the third by problem 6 and the simple observation that $N \lesssim N \times N$, and the last by 7 again. □

**Notes:** This is one of those problems where finding an explicit bijection, or even explicit injections, is a pain, and the most efficient method is to use standard cardinal identities. If you keep studying set theory, you’ll pick up on the best way to approach these problems; the best thing to do is practice.
9. Let $A$ be a set. Prove that there is a set

$$B = \{ \mathcal{P}(y) : y \in A \}.$$  

**Solution:** The trick here is to find an “umbrella set” $C$ where $\mathcal{P}(y) \in C$ for every $y \in A$. From there, a simple application of Selection gives us $B$. Our first impulse is to take $C = \mathcal{P}(A)$, but this doesn’t work; not every subset of $y$ need be an element of $A$, which is what this would require. However, we know that $y \in A$ implies $y \subseteq \bigcup A$, and indeed $z \subseteq \bigcup A$ for every $z \subseteq y$. Thus $z \in \mathcal{P}(\bigcup A)$ for every such $z$, i.e., $\mathcal{P}(y) \subseteq \mathcal{P}(\bigcup A)$. So $\mathcal{P}(y) \in \mathcal{P}(\bigcup A)$, and we take $C = \mathcal{P}(\bigcup A)$. □

**Notes:** One surprisingly common error was to try to use the Pairing axiom to create $B$, which only works if $A$ is itself a pair. Of course, the most natural approach to this problem is to use a Replacement axiom with the formula “$y = \mathcal{P}(x)$”, but you didn’t have that tool during the exam. You could take this problem as a proof of that particular Replacement axiom.

10. Consider two objects $O$ and $I$ (with $O \neq I$), which we will call “pseudosets” or “p-sets”. If $x$ and $y$ are p-sets, say that $x$ is a pseudoelement (p-element) or $y$ (and write $x E y$) iff $y = I$. Which of the axioms of (Zermelo) set theory are true if we interpret “set” and “element” to mean “p-set” and “p-element”? You may skip the axioms of Infinity and Choice.

**Solution:**
- **Existence:** Certainly true. We even have an empty p-set $O$, if you prefer the stronger version of the axiom.
- **Extensionality:** True. We have two p-sets, one has p-elements, the other does not.
- **Selection:** False. There is no p-set $\{O\} = \{x E I : x = O\}$. (Also, if Selection were true, we could use the universal p-set $I$ to create a Russell p-set $B = \{x : x E x\} = \{x \in I : x E x\}$ — thanks to Neil Chua for pointing this out!)
- **Pairing:** False. As noted above, there is no p-set $\{O\} = \{O, O\}$. (Remember, nothing says that $a$ and $b$ in $\{a, b\}$ must be distinct!)
- **Union:** True. $\bigcup O = O$, $\bigcup I = I$.
- **Powerset:** False. There is no $\{O\} = \mathcal{P}(O)$.
- **Foundation:** True. The only nonempty p-set is $I$, and $O E I$ where $O$ and $I$ have no common p-elements.

(For the record, the axiom of infinity is problematic because because the successor operation is not appropriately defined. The axiom of choice is either trivially true or trivially false depending on which variant you prefer.) □
**Bonus:** The following theorem and proof are wrong given the axioms we have studied so far. See if you can find the gap in the proof.

**Theorem.** Let $a$ be any set. Then there is an inductive set $X \ni a$.

Proof: Define $F(n)$ for $n \in \mathbb{N}$ recursively as follows:
1. $F(0) = a$; and
2. $F(S(n)) = S(F(n))$ (where $S(a) = a \cup \{a\}$ as usual).

Now let $X = \mathbb{N} \cup \text{ran}(F)$. $X$ is precisely the sort of set we want. □

**Solution:** (First, the version in the exam contains a misprint; it defines $F(S(n)) = S(a)$, which clearly doesn’t help. Even with the above repair, we have problems.) This scheme doesn’t work because $S$ is not a function! That is to say, it’s not a set of ordered pairs on an appropriate domain, because it’s defined for every set. In order to use the Recursion Theorem, we would need to define a set to which we could restrict $S$; but such a set would have to contain $a$ and be closed under $S$, and that’s almost exactly what we’re trying to show exists. □

In fact, there is no way to prove this theorem without using the Replacement schema and a revamped Recursion Theorem designed to take advantage of “class functions”, as such objects as $S$ are called. For instance, if one assumes all of ZFC and then restricts one’s notion of “set” to the elements of $V_{\omega^2}$—you’ll learn about this soon—all of the axioms of ZFC still hold except for Replacement, and there is no inductive set $X \ni \omega$.  

4