1. Let $S$ be the set of positive integers. Define an order $<'$ on $S$ as follows. Consider two (unequal) positive integers $m$ and $n$. Let $p$ be the largest prime that occurs with different exponents in the prime factorizations of $m$ and of $n$. Order $m$ and $n$ according to which exponent is smaller.

For example, consider $480 = 2^5 \cdot 3 \cdot 5$ and $45 = 3^2 \cdot 5$. Now $3$ is the largest prime that occurs with different exponents in the prime factorizations of $480$ and of $45$. It occurs with exponent $1$ in the prime factorization of $480$ and with exponent $2$ in the prime factorization of $45$. Thus $480 <' 45$.

One can show that $(S,<')$ is a well-ordered set.

(a) Let $n$ be any positive integer. What is the next number after $n$ in the order $<'$?

Solution: The next number (in the order $<'$) after $n$ is $2^n$.

(b) Which ordinal is isomorphic to $(S,<')$? Give the isomorphism if you can.

Here’s how you might figure it out. Let’s make an isomorphism $f$ from an ordinal to $(S,<')$. The smallest number is $1$, so $f(0) = 1$. The next smallest is $2$, so $f(1) = 2$.

In general $f(n) = 2^n$ for $n \in \omega$. The next number after all the powers of $2$ is $3$, followed by $3 \cdot 2$, $3 \cdot 2^2$, and so on. Thus $f(\omega + n) = 3 \cdot 2^n$. The next number after all of those is $3^2$, so $f(\omega \cdot 2) = 3^2$.

Now perhaps the pattern is apparent: $f(\omega^k) = p_k$ where $p_k$ is the $(k+1)$st prime number (in the usual ordering). (Thus $p_0 = 2$, $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, $p_4 = 11$, and so on.) More generally, we let $f$ map

$$\omega^k \cdot n(k) + \omega^{k-1} \cdot n(k-1) + \cdots + \omega \cdot n(1) + n(0)$$

to

$$p_k^{n(k)} p_{k-1}^{n(k-1)} \cdots 3^{n(1)} 2^{n(0)}.$$  

Note that $f$ is an isomorphism from $\omega^n$ to $(S,<')$.

2. Let $\alpha$ be an infinite ordinal and $n$ be a natural number. Prove that $n + \alpha = \alpha$.

Solution: First we prove it for $\alpha = \omega$.

Method 1: we know that $n + \omega = \sup \{ n + k : k < \omega \}$. But $\sup \{ n + k : k < \omega \} = \omega + \{ n + k : k < \omega \} = \omega$.

Method 2: Recall the definition of $n + \omega$. We take a disjoint wosets $(A,<)$ isomorphic to $n$ and $(B,<)$ isomorphic to $\omega$. Let $a_0, a_1, \ldots, a_{n-1}$ and $b_0, b_1, \ldots$ be the elements of $A$ and of $B$ in increasing order. Then

$$f : A \cup B \rightarrow \omega$$

$$f(a_i) = i$$

$$f(b_i) = n + i$$
defines an isomorphism from \((A \cup B, <)\) to \(\omega\). Therefore \(n + \omega = \omega\).

Now we can prove it for any infinite \(\alpha\). Since \(\alpha \geq \omega\), we know that there is a unique \(\beta\) such that \(\alpha = \omega + \beta\). Thus

\[
n + \alpha = n + (\omega + \beta) = (n + \omega) + \beta = \omega + \beta = \alpha.
\]

3. Consider ordinals \(a\) and \(b\) with \(a < b\) and positive integers \(m\) and \(n\). Prove that

\[
\omega^a \cdot m + \omega^b \cdot n = \omega^b \cdot n
\]

Solution: Since \(a < b\), we know that \(a + \beta = b\) for some ordinal \(\beta > 0\). Thus

\[
\omega^a \cdot m + \omega^b \cdot n = \omega^a \cdot (m + \omega^\beta \cdot n) = \omega^a \cdot \omega^\beta \cdot n + \omega^a \cdot n = \omega^{a+\beta} \cdot n \quad \text{by problem 2)}
\]

\[
= \omega^b \cdot n.
\]

4. Find the normal form for \(\alpha + \beta\), where

\[
\alpha = \omega^\omega \cdot 3 + \omega^2 \cdot 5 + 7
\]

\[
\beta = \omega^3 \cdot 9 + \omega \cdot 4 + 13.
\]

Solution. By associativity of addition:

\[
\alpha + \beta = (\omega^\omega \cdot 3 + \omega^2 \cdot 5 + 7) + (\omega^3 \cdot 9 + \omega \cdot 4 + 13)
\]

\[
= \omega^\omega \cdot 3 + (\omega^2 \cdot 5 + (7 + (\omega^3 \cdot 9 + \omega \cdot 4 + 13)) + \omega \cdot 4 + 13)
\]

\[
= \omega^\omega \cdot 3 + (\omega^2 \cdot 5 + \omega^3 \cdot 9 + \omega \cdot 4 + 13)
\]

\[
= \omega^{\omega \cdot 3} + \omega^3 \cdot 9 + \omega \cdot 4 + 13.
\]