MATH 161 HW 8 SOLUTIONS

1. Here’s how you might figure out this problem. Try to find an isomorphism $F$ from an ordinal $\alpha$ to $S$. Now $F(0)$ must be the smallest element of $S$, namely $0 = 1 - 1/1$. Likewise $F(1)$ must be the next smallest, which is $1 - 1/2 = 1/2$. Then $F(2)$ must be the next smallest, which is $1 - 1/3 = 2/3$. In general,

$$F(n) = 1 - 1/(n + 1)$$

for all $n \in \omega$. The next smallest element of $S$ is $2 = 2 - 1/1$, so $F(\omega) = 2 - 1/1$. Then $F(\omega + 1)$ must be the next element of $S$, namely $2 - 1/2$. Perhaps at this point the pattern is evident:

(*)

$$F(\omega \cdot m + n) = (m + 1) - 1/(n + 1).$$

Once you have $F$, you can write the solution to the problem.

Solution: Define $F : \omega^2 \to S$ by (*). We claim that $F$ is an isomorphism, which implies both that $S$ is well-ordered and that $\omega^2$ is the ordinal isomorphic to it. Certainly $F$ is surjective by definition of $S$.

Now suppose $\omega \cdot m + n < \omega \cdot m' + n'$. Then either (1) $m < m'$ or (2) $m = m'$ and $n < n'$. In case (1),

$$(m + 1) - 1/(n + 1) < m + 1 \leq (m' + 1) - 1/(n' + 1)$$

In case (2),

$$(m + 1) - 1/(n + 1) = (m' + 1) - 1/(n + 1) < (m' + 1) - 1/(n' + 1)$$

In either case, we see that

$$F(\omega \cdot m + n) < F(\omega \cdot m' + n').$$

This shows that $F$ is one-one and that $F$ is order-preserving. Hence $F$ is an isomorphism from $\omega^2$ to $S$. \hfill $\square$

2. How you might figure this out: note that in this order, first comes 1 (the only number with no prime factors), then all the primes (in the usual order), then all the products of two primes (in the usual order), then all the products of three primes (in the usual order), and so on. Finally, 0 is the greatest element (in this ordering) because it’s divisible by every prime (and hence the number of prime factors is infinite.) This is like $\omega$ copies of $\omega$, one after the other, followed by one more element (0) at the very end. So $(N, \prec')$ should be isomorphic to $\omega^2 + 1$.

Solution: For $k > 1$, let $A_k$ be the set of natural numbers with exactly $k$ prime factors. Note that $A_k$ is infinite (it includes $p^k$ for every prime $p$), so the map

$$F_k : N \to A_k$$

$$F_k(n) = \text{the } (n + 1)\text{st element of } A_k \text{ (in the usual ordering)}$$

is an isomorphism from $\omega$ to $A_k$. 

1
It’s convenient to adopt a slightly different definition of $A_0$ and $A_1$, since otherwise $A_0$ would have only one element. Thus let us define $A_1$ to be the set of natural numbers with 0 or 1 prime factors. In other words, $A_1$ is 1 together with all the prime numbers. As above, we let $F_1(n)$ be the $(n+1)$st element of $A_1$.

Now define map $g : \omega^2 + 1 \rightarrow \mathbb{N}$ by

$$g(\omega \cdot m + n) = F_{m+1}(n)$$

$$g(\omega^2) = 0$$

Then $g$ is an bijection. Also (as is not hard to check) $g$ is order-preserving (with respect to the usual order on the domain and the order $<'$ of the range.) Thus $F$ is an isomorphism from $\omega^2 + 1$ to $(\mathbb{N}, <')$. □

3. Recall that in the proof of the well-ordering theorem, we used the choice function $g$ to define a map $F$ from ordinals to a set ($\mathbb{N}$ in this problem) by letting $F(0)$ be $g(\mathbb{N})$, $F(1)$ be $g(\mathbb{N} \setminus \{F(0)\})$ and, in general, $F(\alpha) = g(\mathbb{N} \setminus \text{range}(F|\alpha))$ until we run out of elements of $\mathbb{N}$.

In this case, we have

$$F(0) = g(\mathbb{N}) = \text{the seventh smallest element of } \mathbb{N} = 6.$$ 

Likewise

$$F(1) = g(\mathbb{N} \setminus \{F(0)\}) = \text{the seventh smallest element of } \mathbb{N} \setminus \{6\} = 7$$

By (ordinary) induction on $n$, it is easy to prove that for every $n \in \omega$,

$$F(n) = 6 + n.$$ 

Note that we haven’t run out of elements of $\mathbb{N}$. Now

$$F(\omega) = g(\mathbb{N} \setminus \text{range}(F|\omega)) = g(\{0, 1, 2, 3, 4, 5\}) = 0$$

Continuing, we get $F(\omega + 1) = g(\{1, 2, 3, 4, 5\}) = 1$, $F(\omega + 2) = 2$, and (in general) $F(\omega + j) = j$ for $0 \leq j < 6$. Now there are no more elements of $\mathbb{N}$.

Thus in the ordering $<'$, first come the numbers $6, 7, 8, 9, \ldots$ followed by the numbers $0, 1, 2, 3, 4, 5$. The $F$ we have constructed is an isomorphism from $\omega + 6$ to $(\mathbb{N}, <')$.

4. (a) $1 + \omega = \omega + 2 + \omega$, but $1 \neq 2$. (In general $n + \omega = \omega$ for every $n \in \omega$.) (b). Similarly $n \cdot \omega = \omega$ for every nonzero $n \in \omega$. So (for example) $1 \cdot \omega = 2 \cdot \omega$ even though $1 \neq 2$.

(Note: it is easy to prove that $n \cdot \omega = \omega$. Define a map from $n \times \omega \rightarrow \omega$ by

$$f(p, q) = nq + p.$$ 

Then $f$ gives an isomorphism from $(n \times \omega, <)$ to $(\omega, <)$, where the first $<$ is antilexicographical ordering. Hence $n \cdot \omega = \omega$.)

(c) Note that $(1 + 1) \cdot \omega = 2 \cdot \omega = \omega$ (see (b),), but $1 \cdot \omega + 1 \cdot \omega = \omega + \omega \neq \omega$.

5. Part (a) was done in last week’s hw (problem 6 of hw 7).

(b). Claim: $X_n \subseteq T$ for every $n \in \mathbb{N}$.

Proof by induction on $n$: $X_0 = X \subseteq T$ by hypothesis. Now suppose $X_n \subseteq T$. If $a \in X_{n+1} = \cup X_n$, then $a \in b \in X_n$ for some $b$. Since $X_n \subseteq T$, we have $a \in b \in T$. 

Since $T$ is transitive, $a \in T$. We have shown that $a \in X_{n+1}$ implies $a \in T$. Thus $X_{n+1} \subseteq T$. This completes the proof by induction of the claim.

Now since $X_n \subseteq T$ for every $n \in \mathbb{N}$, we have $X' = \bigcup_{n \in \mathbb{N}} X_n \subseteq T$. □

6. (a) Suppose the axiom of foundation is false, i.e. that there a nonempty set $X$ such that $Y \cap X$ is nonempty for every element $Y \in X$.

Let $g$ be a choice function for $\mathcal{P}(X)$. We may assume that $g(\emptyset) \in X$. (Why?)

Now define a function $f : \mathbb{N} \to X$ recursively by

$$f(0) = g(X)$$
$$f(n + 1) = g(f(n) \cap X).$$

Claim: $f(n) \in X$ for all $n \in \mathbb{N}$. Proof of claim by induction: $f(0) \in X$ by definition of "choice function" and because $X$ is nonempty. Now suppose $f(n) \in X$. Then $f(n) \cap X$ is nonempty (by hypothesis), so $f(n + 1) \in f(n) \cap X$ by definition of "choice function". This completes the proof of the claim.

Note that in proving the claim, we also proved that $f(n + 1) \in f(n) \cap X$ for every $n$. In particular, $f(n + 1) \in f(n)$ for every $n$. □

(b) Suppose there is a function $f$ with domain $\mathbb{N}$ such that $f(n + 1) \in f(n)$ for all $n$. Let $X$ be the range of $f$. If $Y \in X$, then $Y = f(n)$ for some $n$, which means that $Y$ is not disjoint from $X$, since $f(n + 1)$ is an element both of $f(n)$ and of $X$. Thus $X$ is a counterexample to the axiom of foundation. □