1. Let \((L, <)\) be a linearly ordered set. Suppose each nonempty subset of \(L\) has a least element and a greatest element. Prove that \(L\) is finite.

**Proof:** since \((L, <)\) and \((\mathbb{N}, <)\) are well-ordered sets, either they are isomorphic, or one is isomorphic to an initial segment of the other. If \((\mathbb{N}, <)\) were isomorphic to a subset of \(L\) (either \(L\) itself or an initial segment), then that subset would not have a greatest element. Thus \((L, <)\) is isomorphic to an initial segment of \((\mathbb{N}, <)\), so \(L\) is finite.

2. Prove that a set \(X\) is transitive if and only if \(X\) is a subset of \(P(X)\) (the power set of \(X\)).

**Proof:** Recall that \(X\) is transitive if and only if every element of an element of \(X\) is also an element of \(X\) or, equivalently, if and only if every element of \(X\) is a subset of \(X\), i.e., if and only if \(X \subseteq P(X)\).

3. Prove that if every \(S \in X\) is transitive, then \(\bigcup X\) is transitive.

**Proof:** Suppose \(a \in b \in \bigcup X\).

Then (by definition of union) \(b \in c\) for some \(c \in X\). Therefore \(c\) is transitive, so (by (*) \(a \in c\). Therefore \(a \in \bigcup X\). We have shown that \(a \in b \in \bigcup X\) implies \(a \in \bigcup X\). Thus \(\bigcup X\) is transitive.

4. Let \((W, <)\) be a well-ordered set. Prove that there is no infinite strictly decreasing sequence in \(W\). Equivalently, suppose \(f : \mathbb{N} \to W\) is any infinite sequence. Prove that \(f\) cannot be strictly decreasing (i.e., prove that there is an \(n\) such that \(f(n + 1) \geq f(n)\)).

**Proof:** Let \(S\) be the range of \(f\). Then \(S\) is a nonempty subset of \(W\), so it has a least element \(a\). By definition of range, \(a = f(n)\) for some \(n \in \mathbb{N}\). Then \(f(n + 1) \geq f(n)\) since \(a\) is the least element of \(S\).

5. You play a game of solitaire as follows. You begin with any number of quarters, dimes, nickels, and pennies. Each day, you are required to exchange one of your coins for any number of coins of smaller denominations. (For example, you could trade one nickel for one billion pennies.) Prove that you must run out of coins in a finite number of days.

**Proof:** consider \(\mathbb{N}^4\) with the lexicographical ordering. (To compare two different elements \(a, b \in \mathbb{N}^4\) we let \(i\) be the first index for which \(a(i) \neq b(i)\). Then we let \(a < b\) if and only if \(a(i) < b(i)\) in the usual ordering of \(\mathbb{N}\).) This ordering is a well-ordering: the proof is essentially the same as in the solution of problem 8 of hw3.
Now on a given day, your coin holdings can be represented by an element $a = (q, n, d, p)$ of $\mathbb{N}^4$: $q$ is the number of quarters, $n$ the number of nickels, and so on. Note that on the next day, you will have a smaller element of $\mathbb{N}^4$. By problem 4, you must run out after a finite number of days.

6. Let $S$ be any set. Prove that there is a transitive set $T$ that contains $S$ as a subset.

Proof: Define a sequence of sets $X_k$ recursively as follows: $X_0 = S$ and $X_{n+1} = \bigcup X_n$. [Note: this uses the axiom of replacement: see theorem 3.6 from chapter 6.] Let $T = \bigcup_{n \in \mathbb{N}} X_n$. (In other words, $T$ is the union of the range of $X$.) Clearly $S = X_0 \subseteq T$. Also, $T$ is transitive. For if $x \in y \in T$, then $y \in X_n$ for some $n$. So $x \in y \subseteq X_n$, so $x \in \bigcup X_n = X_{n+1}$ and therefore $x \in T$.

7. On page 114, do problem 3.3 (but don’t turn it in) and do problem 3.4(a), (b), (c) and (d).

(a) We prove by induction on $n$ that each $V_n$ is finite. First, $V_0 = \emptyset$ is certainly finite. If $V_n$ is finite, then we know $P(V_n) = V_{n+1}$ is finite. Thus every $V_k$ is finite.

Now if $x \in V_\omega = \bigcup_{n \in \mathbb{N}} V_n$, then $x \in V_n$ for some $n$. Since $V_0$ has no elements, $x \in V_n = P(V_{n-1})$ for some $n > 0$. Thus (by definition of powerset) $x \subseteq V_{n-1}$. Since $V_{n-1}$ is finite, this means $x$ is finite.

(b) If $x \in y \in V_\omega$, then $y \in V_n$ for some $n$ so (as in part (a)) $y \subseteq V_{n-1}$. Thus $x \in V_{n-1}$, so $x \in V_\omega$. This proves that $V_\omega$ is transitive.

(d) Prove that $V_n$ is a subset of $V_{n+1}$ for every $n \in \mathbb{N}$.

Proof by induction on $n$: $V_0 = \emptyset$ is a subset of every set, so it is true for $n = 0$. Now suppose it is true for $n$:

$$V_n \subseteq V_{n+1}.$$  

Then every subset of $V_n$ is a subset of $V_{n+1}$, so

$$P(V_n) \subseteq P(V_{n+1}),$$

i.e., $V_{n+1} \subseteq V_{n+2}$.

8. Suppose $X$ is a finite subset of $V_\omega$. Prove that $X \in V_\omega$.

Claim: Let us prove by induction on $k$ that every $k$-element subset of $V_\omega$ is an element of $V_n$ for some $n \in \mathbb{N}$.

The claim is true for $k = 0$, since the only 0-element set is $\emptyset$, which is an element of $V_1$.

Now suppose the claim is true for $k$, and suppose $X$ is a subset of $V_\omega$ with $(k + 1)$ elements. Let $a$ be one of the elements. Then $X' = X \setminus \{a\}$ has $k$ elements, so $X' \in V_n$ for some $n$. Also $a \in V_\omega = \bigcup_{m \in \mathbb{N}} V_m$, so $a \in V_m$ for some $m \in \mathbb{N}$. Let $p$ be the larger of $m$ and $n$. Then (by problem 7(d)), $X = \{a\} \cup X' \subseteq V_p$. Thus $X \in P(V_p) = V_{p+1}$.

9. Mr. Gorgonzola doesn’t believe in any sets except those that are elements of $V_\omega$. Thus he means by the word “set” what we mean by the phrase “element of $V_\omega$.” Which of the axioms of set theory are true under Mr. Gorgonzola’s interpretation of the word “set”?
Existence: true ($\emptyset \in V_1$, so $\emptyset \in V_\omega$)

Extension: true. (It’s true for all sets, so in particular it’s true for those in $V_\omega$.)

Comprehension (i.e., selection): True. If $A \in V_\omega$, $A$ is a finite subset of $V_\omega$ by problem 7(a,b). Thus any subset $B$ of $A$ is a finite subset of $V_\omega$, so by problem 8, $B \in V_\omega$. In particular, if we form the set of $x \in A$ that make a certain sentence $P(x)$ “true”, then $B \in V_\omega$.

[“True” is in quotes because a sentence may be true to us and false for Mr. Gorgonzola and vice versa. For instance, let $A$ be an nonempty element of $V_\omega$, and consider forming the set $B$ of all $x \in A$ that satisfy the sentence following sentence $P(x)$: $x$ is an element of an infinite set. Now for us, $P(x)$ is true for every $x$, because $x \in \{x\} \cup \mathbb{N}$, which is infinite. Thus for us, $B = A$. However, for Gorgonzola, $P(x)$ is false for every $x$ because his universe doesn’t include infinite sets. Thus for him $B$ turns out to be the empty set.]

Pairs: True. If $a, b \in V_\omega$, then $\{a, b\} \in V_\omega$ by problem 8.

Union: True. Note that for any set $A$, $\bigcup P(A) = A$. Now let $x \in V_\omega$. We must show that $\bigcup x \in V_\omega$. Now $x \in V_n = P(V_{n-1})$ for some $n$, so $x \subseteq V_{n-1}$. Thus

$$\bigcup x \subseteq \bigcup (V_{n-1}) = \bigcup (P(V_{n-2})) = V_{n-2}.$$  

SO $\bigcup x \subseteq V_{n-2}$, which implies that $\bigcup x \in V_{n-1} \subseteq V_\omega$.

Power set: True. If $x \in V_\omega$, then $x$ is finite (problem 7a) and $x \subseteq V_n$ for some $n$, so $x \subseteq V_{n-1}$. Thus $P(x) \subseteq P(V_{n-1}) = V_n$, so $P(x) \subseteq V_\omega$.

Infinity: False. Mr. Gorgonzola’s universe contains all the natural numbers as elements, but it does not contain $\mathbb{N}$.

Replacement. True. If $A \in V_\omega$, $A$ is finite. Thus if we replace each element of $A$ by another element in $V_\omega$, the result is a finite subset $B$ of $V_\omega$. By problem 8, $B$ is an element of $V_\omega$.

Foundation. True. Suppose $x$ is a nonempty element of $V_\omega$. Then (by our axiom of foundation) $x$ has an element $y$ that is disjoint from $x$. By transitivity of $V_\omega$, this $y$ is in $V_\omega$ (and therefore exists for Mr. Gorgonzola).

Choice: true. Since $X \in V_\omega$ and since the axioms of pairs, union, and power set are all true for Gorgonzola, it follows that $P(X) \times X$ exists for Gorgonzola, i.e., is an element of $V_\omega$. Thus it is a finite subset of $V_\omega$. Thus any subset of $P(X) \times X$ is a finite subset of $V_\omega$ and thus (by problem 8) is an element of $V_\omega$. In particular, the choice function $f$ is an element of $V_\omega$. 
