1. Define a map

\[ F : \mathcal{P}(\mathbb{R}) \rightarrow C \]

by

\[ F(S) = \bigcup \left\{ \overline{B}(x, 1) : x \in S \right\}. \]

(Here \( \overline{B}(x, y) \) denotes the closed ball of radius \( r \) centered at \( (x, y) \).

By definition of \( C \), the map \( F \) is surjective.

Also, note that \( (x, 1) \in F(S) \) if and only if \( x \in S \). Thus

\[ S = \{ x \in \mathbb{R} : (x, 1) \in F(S) \}. \]

This implies that \( F \) is injective.

Hence \( F \) is a bijection, so \( |C| = |\mathcal{P}(\mathbb{R})| = |2^\mathbb{R}|. \)

2. Suppose \((L, <)\) is a linearly ordered set and \( x, y \in L \). Let us say that \( x \) is the predecessor of \( y \) if (i) \( x < y \) and (ii) there is no \( z \) with \( x < z < y \).

In \( \mathbb{N} \times \{0, 1\}, (0, 0) \) is the only element without a predecessor: the predecessor of \((n, 1)\) is \((n, 0)\), and (if \( n \neq 0 \)) the predecessor of \((n, 0)\) is \((n - 1, 0)\).

However, in \( \{0, 1\} \times \mathbb{N} \), there are two elements without predecessors, namely \((0, 0)\) and \((1, 0)\). To see that \((1, 0)\) has no predecessor, note that if \((a, b) < (1, 0)\), then \( a = 0 \), so \((a, b) = (0, b) < (0, b + 1) < (1, 0)\).

Thus the two linearly ordered sets are not isomorphic.

3. Let \( S \) be a nonempty subset of \( B \). Let \( I_S \) be the set of all \( i \in I \) such that \((i, a) \in S \) for some \( a \). Note that \( I_S \) is nonempty since \( S \) is nonempty. Thus (since \( I \) is well-ordered), \( I_S \) has a least element \( i \).

Now consider the set \( V = \{ a \in A_i : (i, a) \in S \} \). Since \( i \in I_S \), the set \( V \) is nonempty. Thus \( V \) has a least element \( v \).

Now it is easy to show that \((i, v)\) is the least element of \( S \).

(Proof: Let \((j, w) \in S \). We must show that \((i, v) \leq (j, w) \). Since \((j, w) \in S \), \( j \in I_S \), so \( i \leq j \) (since \( i \) is the least element of \( I_S \)). If \( i < j \), then \((i, v) < (j, w) \), so we are done. Thus suppose \( i = j \). Since \((j, w) = (i, w) \in S \), \( w \in V \) (by definition of \( V \)). Thus \( v \leq w \) since \( v \) is the least element of \( V \). Thus \((i, v) \leq (i, w) = (j, w) \). \( \square \)