1. Suppose \( a < b \). We prove by induction on \( c \) that \( a + c < b + c \).

Note \( a + 0 = a < b = b + 0 \) so \( a + 0 < b + 0 \). Thus the assertion is true for \( c = 0 \).

Suppose it is true for \( c = n \):

\[
a + n < b + n.
\]

Then

(*) \[
(a + n) + 1 \leq b + n.
\]

Now \( x < x + 1 \) for all \( x \) (by definitions of successor and of \(<\) ). In particular, \( b + n < (b + n) + 1 \). Combining this with (*) gives

\[
(a + n) + 1 < (b + n) + 1,
\]

or (equivalently) \( a + (n + 1) < b + (n + 1) \). So the assertion is true for \( y = n + 1 \).

This completes the proof by induction.

2. There are many ways to prove this. Here are two:

First solution (by contradiction): Suppose there is an \( x \geq n \) for which \( P(x) \) is false. Then the set

\[
S = \{ x \in \mathbb{N} : x \geq n \text{ and } P(x) \text{ does not hold} \}
\]

is nonempty. Thus it has a least element \( k \). Since \( k \in S \), \( k \geq n \). Since \( P(n) \) holds, \( k \neq n \), so \( k > n \). Thus \( k \neq 0 \), so \( k = j + 1 \) for some \( j \) (see problem 1 of hw 3). Since \( k > n, j \geq n \). Now \( P(j) \) must be true, since otherwise \( j \) would be in \( S \), which is impossible since \( j + 1 \) is the least element in \( S \). Since \( j \geq n \) and \( P(j) \) is true, \( P(j + 1) \) is true. But that is impossible since \( j + 1 (= k) \) is an element of \( S \).

Second solution: If \( n = 0 \), this is just ordinary induction. So we may assume that \( n > 0 \). Let \( Q(x) \) be the statement:

\[
x < n \text{ or } P(x).
\]

We prove by induction that \( Q(x) \) holds for every natural number \( x \).

Note \( Q(0) \) is true because \( 0 < n \).

Now suppose \( Q(x) \) is true. Note \( x < n \) or \( x \geq n \). We handle these two cases separately.

Case (i): \( x < n \). Then \( x + 1 \leq n \). That is, \( x + 1 < n \) or \( x + 1 = n \). If \( x + 1 = n \), then \( P(x + 1) \) is true. Thus \( x + 1 < n \) or \( P(x + 1) \). That is, \( Q(x + 1) \) holds.

Case (ii): \( x \geq n \). Since \( Q(x) \) holds, this implies \( P(x) \) is true. Since \( x \geq n \) and \( P(x) \) is true, \( P(x + 1) \) is true. Thus \( Q(x + 1) \) is true.

So \( Q(0) \) is true, and \( Q(x) \) implies \( Q(x + 1) \) for every natural number \( x \). Thus by induction, \( Q(x) \) is true for all \( x \in \mathbb{N} \).
3. Note $x = x + 0$ (this is part of the definition of addition), so $x \leq x + 0$. Thus the assertion is true for $y = 0$. Thus we may suppose $0 \neq y$. Then $0 < y$. Hence $x + 0 < x + y$ by problem 1 (and commutativity of addition). Thus $x < x + y$ (since $x + 0 = x$).

We can also prove it by induction on $y$. The case $y = 0$ is handled exactly as the previous paragraph. Now suppose it is true for $y = n$;

(*)

$x \leq x + n$.

Now $x + n < (x + n) + 1$ (because every set is an element of its successor) and $(x + n) + 1 = x + (n + 1)$, so by (*)

$x < x + (n + 1)$.

This completes the proof by induction.

4. Fix a natural number $m$, and let $P(n)$ be the statement:

There is a $k \in \mathbb{N}$ such that $m + k = n$.

We prove by induction that $P(n)$ holds for all $n \geq m$. (See problem 2.)

Note $P(m)$ is true because $m + 0 = m$.

Suppose $P(n)$ is true, i.e., that

$m + k = n$

for some $k$. Adding 1 to both sides (and using associativity)

$m + (k + 1) = n + 1$

So $P(n + 1)$ is true. This completes the proof by induction.

5(1). Proof by induction on $z$.

The case $z = 0$: $x(y + 0) = xy = xy + 0 = xy + x0$, so it is true for $z = 0$.

Now suppose it’s true for $z$. Then

$x(y + (z + 1)) = x((y + z) + 1)$

$= x(y + z) + x$ (def of multiplication)

$= (xy + xz) + x$ (true for $z$)

$= xy + (xz + x)$

$= xy + x(z + 1)$ (def of mult.)

so it’s true for $z + 1$. This completes the proof by induction.

5(2). 

$x \cdot 1 = x(0 + 1) = x \cdot 0 + x$ (def of mult.)

$= 0 + x$ (def of mult.)

$= x$ □

6. Let use F-set, F-element, etc to denote Fonebone’s understanding of set, element, etc. Thus the F-sets are the integers, $x$ is an F-element of $y$ if and only if $x + 1 = y$, etc. So we consider each axiom:

F-Existence: There is an F-set. True.
F-Extension: Two F-sets are equal if they have the same F-elements. True. (Suppose $x$ and $y$ are F-sets with the same F-elements. Then since $x - 1$ is an F-element of $x$, it’s also an F-element of $y$. Therefore $(x - 1) + 1 = y$, i.e. $x = y$.)

F-Selection: False. From F-existence and F-extension we can prove the existence of an F-set with no F-elements. But every F-set $x$ has an F-element, namely $x - 1$.

F-Pairs. FALSE. Every F-set has exactly one F-element. Thus there is no F-set containing both 0 and 1 (for example) as F-elements.

F-union TRUE. In fact, the strong form of the axiom is true. We need to show, for each F-set $a$, that there is an F-set $u$ such that

(*) $x \in_F u$ if and only if $x \in_F y$ and $y \in_F a$ for some F-set $y$.

There is one and only one F-element of $a$, namely $a - 1$, so (*) is equivalent to:

$x \in_F u$ if and only if $x \in_F a - 1$

Of course this is true for $u = a - 1$. Thus the F-union of $a$ is $a - 1$.

(Warning: even though the axiom of union is true, $a \cup_F b$ does not exist (for Fonebone) unless $a = b$. This is not a contradiction. To prove existence of $a \cup b$, we used the axiom of union and the axiom of pairs. As we saw, only the former is true for Fonebone.)

F-Powerset axiom. This is TRUE (even in the strong form). Note that each F-set $x$ has exactly one F-subset, namely $x$ itself. Thus $x + 1$ is an F-set whose F-elements are precisely the F-subsets of $x$.

F-infinity. This doesn’t make much sense because there is no F-emptyset.

F-Foundation is true. Every F-set $x$ has an F-element, namely $x - 1$, that is F-disjoint from $x$. (Of course $a$ and $b$ are F-disjoint means that there is no $c$ that is an F-element of $a$ and of $b$.)

6. There are many ways of doing this. Here is one. Remove apple $(p, k)$ at stage $N = 2^p(2^k + 1) - 1$.

Equivalently, to determine which apple to discard at stage $N$, let $2^p$ be the largest power of 2 that divides $N + 1$. Then $(N + 1)/(2^p)$ will be on odd number $2k + 1$. Discard apple $(p, k)$.

Clearly every apple gets discarded. We need to check, however, that the rule does not ask us to discard an apple we haven’t yet received! Note that

$2^p(2k + 1) - 1 \geq (2k + 1) - 1 = 2k \geq k$

as needed.

Here’s another way to do it. At stage $2^p3^k$, discard apple $(p, k)$ if it hasn’t already been discarded. Otherwise discard any remaining apple. At each other stage, choose any remaining apple to discard.