1. Let $P(n)$ be the statement: $n = 0$ or $n$ is the successor of some number. We prove it by induction on $n$.

Since $0 = 0$, the statement $P(0)$ is true.

Now suppose $P(n)$ is true. Then $S(n)$ is the successor of some number (namely $n$), so $P(S(n))$ is true. Hence $P(n)$ implies $P(n + 1)$.

This completes the proof by induction that $P(n)$ holds for all $n$.

2. Suppose $S(x) = S(y)$. Then (exactly as in last week’s hw) $x \in y$ and $y \in x$. (That part of the argument didn’t use the axiom of foundation.) But we proved in class (this is also in the text) that for natural numbers $x$ and $y$ we cannot have both $x \in y$ and $y \in x$. (This was part of proving that $\in$ gives a strict linear ordering of the natural numbers.)

3. Suppose $A \neq N$. Then $\{x \in N : x \notin A\}$ contains a least element $e$. We claim that $e = A$. We prove it by the axiom of extension.

Suppose $x \in e$. Then $x$ is a natural number and $x < e$. Thus $x \in A$ (since $e$ is the least natural number not in $A$.) This proves $e \subseteq A$.

Now suppose $x \in A$. Then by transitivity of $A$, every natural number $\leq x$ is an element of $A$. Since $e \notin A$, this means $e$ cannot be $\leq x$. Thus $x < e$ or (equivalently) $x \in e$. This proves that $A \subseteq e$.

Since $e \subseteq A$ and $A \subseteq e$, $A = e$ by the axiom of extension.

4. Let $C$ be a set with the given property. By problem 3, we need only show that $C$ is transitive, i.e., that

(*) $x < y$ and $y \in C$ imply $x \in C$.

Let us prove (*) by induction on $y$. For $y = 0$, it is vacuously true (since there are no $x$ with $x < 0$.)

Suppose it is true for $y = n$. Now if $x < y + 1$ and $y + 1 \in C$, then

(i) $x \leq n$ and

(ii) $y \in C$

(by assumption on $C$.) If $x = y$, then $x \in C$ (by (ii)). If $x \neq y$, then $x < y$ (by (i)), so $x \in C$ by (ii) and the induction hypothesis (*). Either way $x \in C$. This shows that (*) is true with $y + 1$ in place of $y$.

Thus by induction (*) holds for all natural numbers $x$ and $y$.

5. Let $f : N \to N$ be any function. We will prove that $f(k + 1) \geq f(k)$ for some $k \in N$. Let $R$ be the range of $f$. Then $R$ is not empty (since, for example, $f(0)$
is an element). We proved in class that every nonempty subset of $\mathbb{N}$ has a least element, so $R$ has a least element $x$. By definition, $x = f(k)$ for some $k \in \mathbb{N}$. Since $f(k + 1) \in R$, $f(k + 1) \geq f(k)$.

6. Let $f$ be any function with domain $\mathbb{N}$. We will prove that there is a $k \in \mathbb{N}$ for which $f(k + 1) \notin f(k)$. Let $R$ be the range of $f$. By the axiom of foundation, there is an element $x \in R$ that is disjoint from $R$:

(by definition of $Y$ of least element $\bar{x}$, $f < g$. Thus $x \in R$ is an element, so

$\bar{x} \in R$. By definition, also, $f(k + 1) \notin f(k)$.

7. Suppose $f, g : \mathbb{N} \rightarrow \mathbb{N}$ and $f \neq g$. Then $f(i) \neq g(i)$ for some $i$, so the set

is nonempty. Thus it has a least element $k = k(f, g)$. Note that this is the only number $k$ with the property

Since the numbers $f(k)$ and $g(k)$ are not equal, one, say $f(k)$ is smaller: $f(k) < g(k)$. Thus $f < g$. By definition. Also, $g(k) \neq f(k)$, so $g \neq f$. This proves that (for $f \neq g$, $f < g$ or $g < f$, but not both.)

Now suppose $f < g$ and $g < h$. Let $k_1 = k(f, g)$ and $k_2 = k(g, h)$.

Case 1: $k_1 < k_2$. Then $f(i) = g(i) = h(i)$ for $i < k_1$ and

so $f < h$.

Case 2: $k_1 = k_2$. Then $f(i) = g(i) = h(i)$ for $i < k_2$ and

so $f < h$.

Case 3: $k_1 > k_2$. Then $f(i) = g(i) = h(i)$ for $i < k_2$ and

so $f < h$.

8. Let

$X = \{x \in \mathbb{N} : (x, y) \in A \text{ for some } y \in \mathbb{N}\}$.

Then $X$ is a subset of $\mathbb{N}$ and $X$ is nonempty since $A$ is nonempty. Thus $X$ has a least element $\bar{x}$.

Now let

$Y = \{y \in \mathbb{N} : (\bar{x}, y) \in A\}$.

Note that $Y$ is nonempty since $a \in X$. Thus $Y$ has a least element $\bar{y}$. By definition of $Y$, $(\bar{x}, \bar{y}) \in A$.

Now let $(x, y)$ be an element of $A$. We must show that $(\bar{x}, \bar{y}) \leq (x, y)$. Then $x \in X$ (by definition of $X$.) Thus

$x \leq x$

If $\bar{x} < x$, then $(\bar{x}, \bar{y}) < (x, y)$, so we are done.
Thus suppose

(i) \( \bar{x} = x. \)

Then \((\bar{x}, y) \in A, \) so \( y \in Y \) (by definition of \( Y \)). Thus

(ii) \( \bar{y} \leq y \)

since \( \bar{y} \) is the least element of \( Y \). Hence \((\bar{x}, \bar{y}) \leq (x, y)\) by (i) and (ii).

9. Let the apples received at stage \( n \) be numbered

\[ 10n, 10n + 1, \ldots, 10n + 9 \]

Let \( x_n \) be the apple removed at stage \( n \). Note that \( < x_0, x_2, \cdots > \) can be any sequence of natural numbers such that

1. \( x_n \leq 10n + 9 \) (you can’t remove an apple until you get it), and
2. \( x_n \neq x_m \) for \( n \neq m \). (You can’t remove the same apple twice.)

Of course the apples we’re left with at the end are those with numbers in \( \mathbb{N} \setminus \text{ran}(x) \).

If \( x_n = n \) for all \( n \) (at each stage \( n \), we remove apple number \( n \)), then at the end we have no apples.

If \( x_n = 10n \) for all \( n \), then at the end we have infinitely many apples (corresponding to all numbers not divisible by 10.)