1. Suppose $A$ is a set. Let $B = \{x \in A : x \notin x\}$. [Of course $B$ exists and is unique by the axioms of selection (comprehension) and extension.] Prove that $B$ is not an element of $A$.

Note: This shows that there is no “universal set” (i.e., set that contains everything).

**Proof.** By definition of $B$,

$$x \in B \text{ if and only if } x \in A \text{ and } x \notin x.$$ 

This holds for every $x$, so in particular it holds for $x = B$:

$$B \in B \text{ if and only if } B \in A \text{ and } B \notin B.$$ 

Thus if $B$ were an element of $A$, we would have:

$$B \in B \text{ if and only if } B \notin B,$$

which is a contradiction. Thus $B$ cannot be an element of $A$. □

2. Let $A$ be a set. Prove that $\{x : x \notin A\}$ can not be a set.

**Proof.** Suppose it were a set. Call that set $B$. Then the union $A \cup B$ would contain everything. (Because given any $x$, either $x \in A$ or $x \notin A$, i.e., either $x \in A$ or $x \in B$. Either way, $x \in A \cup B$.) But as we have seen (for instance in problem 1), there is no universal set. □

3. A set $S$ is called a singleton provided $S = \{a\}$ for some $a$. Prove that there does not exist a set $S$ that contains every singleton. Hint: prove that if there were such a set $S$, then there would exist a “universal” set.

**Proof.** Suppose there is such a set $S$. For every $x$ the set $\{x\}$ is a singleton and is therefore an element of $S$. (We know that $\{x\}$ exists by the axiom of pairs.) Since $x$ is an element of $\{x\}$ which is an element of $S$, $x$ is an element of $\bigcup S$ (which is a set by the axiom of union.) Thus $\bigcup S$ is a set that contains every $x$. But we already know that there is no such universal set. □

4. Let $S(x)$ be a sentence. Suppose there are some $x$’s for which $S(x)$ is true. Prove that

(*) \{y : y \text{ is an element of } x \text{ for every } x \text{ such that } S(x) \text{ is true}\}

is a set.

**Proof.** Let $A$ be a set for which $S(A)$ is true. Then (*) is the same as

\{y \in A : y \in x \text{ for every } x \text{ such that } S(x) \text{ is true}\}.

This set exists by the axiom of selection (a.k.a. comprehension). □
Note: In the solution to this problem, one cannot use \[ \{x : P(x)\} \], because (depending on what the sentence \( P(x) \) is) such a set might not exist.

5. Let \([a, b]\) be the set \( \{a, \{b\}\} \). Explain why this would not be a good definition of ordered pair.

Solution: For example, let \( a = \{\emptyset\} \) and \( b = \{a\} \). Note that \( \emptyset, a, \) and \( b \) are all different by the axiom of extension. Now \([a, a] = \{a, \{a\}\} = \{a, b\}\)

and \([b, \emptyset] = \{b, \{\emptyset\}\} = \{b, a\}\).

By the axiom of extension, these two sets are equal: \([a, a] = [b, \emptyset]\)

However, \( a \neq b \) and \( a \neq \emptyset \).

(If \([\cdot, \cdot]\) were a good definition of ordered pair, then \([x, y]\) would equal \([z, w]\) if and only if \( x = z \) and \( y = w \).)

6. Prove that
\[
A \times (B \cup C) = (A \times B) \cup (A \times C)
\]
for all sets \( A, B, \) and \( C \).

Proof. Let \( z \) be an element of the LHS (i.e., left hand side) of the equation. Then \( z = (a, y) \) for some \( a \in A \) and some \( y \in (B \cup C) \). Since \( y \) is in \((B \cup C)\), \( y \) is in \( B \) or \( y \) is in \( C \). In the first case, \((a, y)\) is in \( A \times B \). In the second case, \((a, y)\) is in \( B \times C \). Either way, \( z = (a, y) \) is in their union, i.e. the RHS of the equation. This proves that every element of the LHS is also an element of the RHS.

Now let \( z \) be an element of the RHS. Then \( z \) is in \( A \times B \) or in \( A \times C \). In the first case, \( z = (a, y) \) for some \( a \in A \) and for some \( y \in B \). In the second, \( z = (a, y) \) for some \( a \in A \) and for some \( y \in C \). Either way, \( y \) is in \((B \cup C)\). Thus \( z = (a, y) \) is in \( A \times (B \cup C) \). This proves that every element of the RHS is also an element of the LHS.

By the axiom of extension, the two sets (the LHS and the RHS) are equal. \( \square \)

7. Give an example of each of the following:

7(a). A relation that is reflexive and symmetric but not transitive. Example:
\[
\{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| \leq 1\}
\]

7(b). A relation that is reflexive and transitive but not symmetric. Example:
\[
\{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq y\}
\]

7(c). A relation that is symmetric and transitive but not reflexive.

The simplest example is as follows. Let \( X \) be a nonempty set. Then the emptyset is a relation on \( X \) that is symmetric and transitive, but not reflexive. \( [p] \) [You don’t have to limit yourself to things we’ve proved to exist. Your examples can use integers, real numbers, or whatever.]
More generally, suppose $A$ is a subset of $X$ that is not equal to $X$, so that $X$ contains some element $p$ not in $A$. Consider the relation

$$R = \{(z, w) : z \text{ and } w \text{ are both elements of } A\}.$$ 

Then $R$ is symmetric and transitive, but it is not reflexive on $X$ (since $(p, p) \notin R$.)