HW #6 - Solutions of The Rest of The Problems

Note: Whenever I say only one eigenvector, I mean (of course) up to scaling.

6.5.11

The system is:

\[ \dot{x} = y = f_1(x, y) \]
\[ \dot{y} = -by + x - x^3 = f_2(x, y) \]

The critical points occur at \( x = 0, x(1 - x^2) = 0 \) which gives us \( x = 0, \pm 1 \)

We need to linearize by Taylor’s f-la, so we compute derivatives

\[
\begin{pmatrix}
  f_{1x} & f_{1y} \\
  f_{2x} & f_{2y}
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 \\
  1 - 3x^2 & -b
\end{pmatrix}
\]

Which leads to the following linearizations:

1) at \((0, 0)\)

\[
\begin{pmatrix}
  \dot{x} \\
  \dot{y}
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 \\
  1 & -b
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]

In this case we get eigenvalues \( \lambda = -b/2 \pm 1 \) and corresponding eigenvectors \((1, -b/2 \pm 1)\) and for \( b \) positive close to 0 this gives a saddle point.

2) at \((1, 0)\) and \((0, -1)\) we get identical linearizations

\[
\begin{pmatrix}
  \dot{x} \\
  \dot{y}
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 \\
  -2 & -b
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]

The eigenvalues are \(-b/2 \pm i\sqrt{2}\) and so we get stable spirals both with a counterclockwise direction.

After drawing the diagram carefully (Matlab may help) we see that \((1, 0)\) has a basin of attraction which is a bounded set. The reason is that outside of this bounded set the system has closed orbits. All this can be seen in the following Matlab output.
6.8.6

Notice that each node spiral and center has index +1 while each saddle has index -1. So by Theorems 6.8.1 and 6.8.2 $N + F + C - S = +1$ as desired.

6.8.7 See back of the book.

6.8.8

a)

b) We know $I_{c1} = I_{c2} = I_{c3} = 1$ since these are closed orbits. Suppose that there were no fixed points in the intermediate region. Then by theorem 6.8.1 we would get $I_{c3} = I_{c2} + I_{c1} = 1 + 1 = 2$ which is a contradiction. So there are fixed points in this region.

7.1.3

Stable cycle at $r=1$ and unstable at $r=2$. Rotation changes direction at $r = \sqrt{2}$. 