HW #4 - Solutions of Selected Problems

5.1.9 (Page 141)

b) Follow the hint:

\[ x \dot{x} - y \dot{y} = x(-y) - y(-x) = 0. \]

Notice that \((x^2)' = 2x\dot{x}\), \((y^2)' = 2y\dot{y}\), integrating the above equation we obtain that

\[ x^2 - y^2 = C, \]

where \(C\) is a constant.

c) The system can be written as

\[ \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \]

The matrix has \(\tau = 0\) and \(\Delta = -1\), so the characteristic equation is \(\lambda^2 - 1 = 0\). Hence

\[ \lambda_1 = 1, \quad \lambda_2 = -3. \]

Next we find the eigenvectors. Given an eigenvalue \(\lambda\), the corresponding eigenvector \(v = (v_1, v_2)\) satisfies

\[ \begin{pmatrix} -\lambda & -1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

For \(\lambda_1 = 1\), this yields \(\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\), which has a nontrivial solution \((v_1, v_2) = (1, -1)\). All its scalar multiples form the unstable manifold \(x = -y\).

Similarly, for \(\lambda_2 = -1\), there is a nontrivial solution \((v_1, v_2) = (1, 1)\). Its scalar multiples form the stable manifold \(x = y\).

d) Let \(u = x + y\), \(v = x - y\), then

\[ \dot{u} = \dot{x} + \dot{y} = -y - x = -u. \]

Similarly \(\dot{v} = v\). For an arbitrary initial condition \((u_0, v_0)\), the solution is

\[ u(t) = u_0 e^{-t} \]
\[ v(t) = v_0 e^t \]

e) Notice that 0 is stable for \(\dot{u} = -u\), but unstable for \(\dot{v} = v\). Therefore to get the stable manifold, we must have \(v = 0\). For the same reason to get the unstable manifold we must have \(u = 0\).
f) The initial condition in terms of $u$ and $v$ is $(x_0 + y_0, x_0 - y_0)$, therefore the general solution is

$$x(t) = \frac{1}{2}(u(t) + v(t)) = \frac{1}{2}((x_0 + y_0)e^{-t} + (x_0 - y_0)e^t)$$

$$y(t) = \frac{1}{2}(u(t) - v(t)) = \frac{1}{2}((x_0 + y_0)e^{-t} - (x_0 - y_0)e^t).$$

5.1.10 (Page 141)

a) The coefficient matrix \[
\begin{pmatrix}
0 & 1 \\
-4 & 0
\end{pmatrix}
\] has $\tau = 0$, so the origin is a center, it’s Liapunov stable.

c) In this system $x$ is never changed. If we start at a point $(x_0, y_0)$ where $x_0$ is positive, then the solution of $\dot{y} = x_0$ is $y = x_0t$, which goes arbitrarily large in the long run. So in this case the origin doesn’t have any type of stability.

e) Notice that the system is already decoupled. Since 0 is a stable fixed point for both systems of $x$ and $y$, an arbitrary flow starting at any point always gets closer to the origin as time evolves, so it’s asymptotically stable.

5.2.3 (Page 143)

The coefficients matrix is \[
\begin{pmatrix}
0 & 1 \\
-2 & -3
\end{pmatrix}
\], which has $\tau = -3$, $\Delta = 2$. The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$, hence

$$\lambda_1 = -1, \quad \lambda_2 = -2.$$

Similar to the above, the corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$
The origin is a stable node.

5.2.5 (Page 143)

The coefficients matrix is \( \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \), which has \( \tau = 2, \Delta = 1 \). The characteristic equation is \( \lambda^2 - 2\lambda + 1 = 0 \), hence

\[ \lambda_1 = \lambda_2 = 1. \]

An eigenvector \( v = (v_1, v_2) \) satisfies

\[ \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]

which has a nontrivial solution \( (v_1, v_2) = (2, 1) \). Since there is only one eigenvector, the origin is a degenerate node.

5.2.7 (Page 143)

The coefficients matrix is \( \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix} \), which has \( \tau = 0, \Delta = 9 \). The characteristic equation is \( \lambda^2 + 9 = 0 \), hence there is no real eigenvalue. According to Figure 5.2.8 in the textbook, the origin is a center.
The coefficients matrix is \[
\begin{pmatrix}
4 & -3 \\
8 & -6
\end{pmatrix}
\], which has \(\tau = -2, \Delta = 0\). The characteristic equation is \(\lambda^2 + 2\lambda = 0\), hence
\[
\lambda_1 = 0, \quad \lambda_2 = -2.
\]

The corresponding eigenvectors are \(v_1 = \left(\begin{array}{c}3 \\ 4\end{array}\right)\), and \(v_2 = \left(\begin{array}{c}1 \\ 2\end{array}\right)\). Since \(\Delta = 0\) and \(\tau < 0\), the origin is a non-isolated fixed point.

5.3.2 (Page 144)

a) According to the system: Romeo tends to echo Juliet. Juliet gets excited by her own affectionate feeling but wants to run away the more Romeo loves her.
b) The coefficients matrix is \( \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \), which has \( \tau = 1, \Delta = 1 \). The characteristic equation is \( \lambda^2 - \lambda + 1 = 0 \), hence there are two different complex eigenvalues. Since \( \Delta > 0 \), \( \tau > 0 \), and \( \tau^2 - 4\Delta < 0 \), the origin is an unstable spiral. This implies that unless initially they are unrelated \( (R = J = 0) \), their story will continue forever.

c) The graph can be plotted using Matlab.

At the end of this document you can find the code for the above picture.

5.3.6 (Page 144)

We first get rid of the uninteresting case \( a = b = 0 \). In this case the way they feel about each other is never changed. In the following we assume that one of \( a \) and \( b \) is nonzero. (so we can claim the second eigenvector is \((0,1)\), see below)

The coefficients matrix is \( \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \), which has \( \tau = b, \Delta = 0 \). The characteristic equation is \( \lambda^2 - b\lambda = 0 \), hence

\[ \lambda_1 = 0, \quad \lambda_2 = b. \]

The corresponding eigenvectors are

\[ \mathbf{v}_1 = \begin{pmatrix} b \\ -a \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Since \( \Delta = 0 \), there are always non-isolated fixed points which form the subspace spanned by vector \( \begin{pmatrix} b \\ -a \end{pmatrix} \).

If \( b < 0 \), all flows approach the line \( ax + by = 0 \) vertically. If \( b > 0 \) they tend to leave the line, still in the opposite direction. If \( b = 0 \), both eigenvectors are \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), the (Liapunov) stability of the fixed points on \( x = 0 \) depends on the sign of \( a \). The vector fields are shown as follows.
\[ b > 0 \]
\[ b < 0 \]
\[ b = 0, a > 0 \]
\[ b = 0, a < 0 \]
Appendix: Matlab code for graph in 5.3.2

function rnj

    % t goes from 0 to 8
    % the initial conditions are 1 and 0
    tspan = [0 8];
    y0 = [1; 0];

    % call ODE solver "ode45" to solve the system
    % determined by f (see the end of the code), tspan
    % and y0
    [t, y] = ode45(@f, tspan, y0);

    % the solutions are returned to variable y
    % y(:,1) and y(:,2) are the data for R(t) and J(t)
    plot(t, y(:, 1), '-o', t, y(:, 2), '-.');

    % add legend
    legend('Romeo', 'Juliet', 2);

    % set the background color to white
    set(gcf, 'Color', 'w');

    % the system: y(1) is R, y(2) is J
    function dydt = f(t, y)
    dydt = [y(2) - y(1) + y(2)];

    end