Solutions 1

2.2.5 There are no equilibria: $\frac{1}{2} \leq f(c) \leq \frac{3}{2}$, so $x(t)$ always moves to the right with speed between 1/2 and 3/2.

2.2.7 For $x > 0$, $e^x > 1 \geq \cos x$, so $f(x) = e^x - \cos x > 0$ has no equilibria in this range.

Note $f(0) = e^0 - \cos 0 = 1 - 1 = 0$, so 0 is an equilibrium. Also, $f'(0) = e^0 + \sin 0 = 1$, so it is a source (or unstable equilibrium.)

For $x < 0$, $\cos x$ oscillates between $-1$ and 1, but $e^x$ stays between 0 and 1, so the two graphs must cross infinitely many times. Thus $f(x)$ has infinitely many 0’s. Note “upcrossings” of $e^x$ by $\cos x$ alternate with downcrossings, so stable and unstable equilibria alternate.

As $x \to -\infty$, $e^x \to 0$, so the equilibria will get closer and closer to the zeroes of $\cos x$, that is, close to the odd multiples of $\pi/2$.

(See the graph on the webpage.)

2.2.13 (Page 38)

Part a) We need to solve

$$\frac{mdv}{mg-kv^2} = dt.$$

Notice that

$$\frac{m}{mg-kv^2} = \frac{1}{2g} \left( \frac{1}{1-\sqrt{\frac{k}{mg}}v} + \frac{1}{1+\sqrt{\frac{k}{mg}}v} \right), \quad \text{(a little tricky)},$$

so we get the solution

$$\sqrt{\frac{m}{4gk}} (\log \frac{1+\sqrt{\frac{k}{mg}}v}{1-\sqrt{\frac{k}{mg}}v}) = t + C.$$

Putting in the condition $v(0) = 0$ we get $C = 0$. Therefore the analytical solution is

$$v = \frac{rm}{k} (e^{rt} - e^{-rt})^\frac{1}{2}, \quad \text{where} \quad r = \sqrt{gk/m}.$$

As this course does not emphasize on solving ODE, you can just solve it using some math software such as Matlab, Mathematica, etc.

Part b) When $t \to \infty$, both $e^{-rt}$ terms in the above vanish and the big fraction becomes 1. The limit is whatever remains, which turns out to be $(rm)/k = \sqrt{mg/k}$.

Part c) Now we solve it geometrically. The equation can be written as

$$\dot{v} = g - (k/m)v^2,$$
and we set it equal to 0. The graph of \( \dot{v} \) versus \( v \) is a parabola crossing the \( x \)-axis from the above. The terminal velocity is the stable fixed point \( v = \sqrt{mg/k} \).

2.3.2 Note that \( \dot{x} = f(x) = x(k_1a - k_{-1}x) \), so the equilibria are at 0 and at \( x = (k_1a)/(k_{-1}) \). Note also that \( f(x) > 0 \) for \( x \) between these two equilibria and is \( < 0 \) for \( x \)'s outside that interval. Thus 0 is unstable and the other equilibrium is stable.

2.4.2 (Page 40)

The fixed points are 0, 1 and 2. Since \( f'(x) = x(x - 1) + x(x - 2) + (x - 1)(x - 2) \), we have

\[
\begin{align*}
    f'(0) &= 2, \quad \text{0 is unstable}, \\
    f'(1) &= -1, \quad \text{1 is stable}, \\
    f'(2) &= 2, \quad \text{2 is unstable}.
\end{align*}
\]

2.4.8 (Page 40)

Letting \( \dot{N} = 0 \) we get \( N = 1/b \). Taking derivative:

\[
f'(N) = -aln(bN) - \frac{a}{b},
\]
then \( f'(1/b) = -\frac{a}{b} < 0 \), 1/b is stable.

2.4.9 (Page 40)

Part (a) Separate variables and integrate to get a solution

\[
t + C = -\int dx \frac{1}{x^3} = \frac{1}{2x^2}.
\]
Therefore \( x(t) = \sqrt{\frac{1}{2(t+C)}} \), where \( C \) is determined by initial condition. Whatever the \( C \) is, the whole thing inside square root goes to 0 when \( t \to \infty \), so \( x(t) \to 0 \), but only at the speed of \( t^{1/2} \), much slower than exponential.

(See the graph on the webpage.)

2.5.2 (Page 40) Suppose \( x(t) \) reaches \( \infty \) in time \( T \). From the equation \( \frac{dx}{dt} = f(x) \) we see that \( dt = \frac{dx}{f(x)} \), so

\[
T = \int_0^T dt = \int_{x_0}^\infty \frac{dx}{f(x)}.
\]
In this problem, \( f(x) = 1 + x^{10} \), so

\[
T = \int_{x_0}^\infty \frac{dx}{1 + x^{10}}.
\]
If \( x_0 \geq 1 \),
\[
T = \int_{x_0}^{\infty} \frac{dx}{1 + x^{10}} \leq \int_{x_0}^{\infty} \frac{dx}{1 + x^2} = (\arctan x)|_{x_0}^\infty = \frac{\pi}{2} - \arctan(x_0) < \infty.
\]
If \( x_0 < 1 \), then
\[
T = \int_{x_0}^{1} \frac{dx}{1 + x^{10}} + \int_{1}^{\infty} \frac{dx}{1 + x^{10}}.
\]
We already showed that the second integral is finite. But the first integral is less than \( 1 - x_0 \) (since the integrand is always less than 1). So \( T < \infty \).

2.5.3 (Page 40)

Since the function \( f(x) = rx + x^3 \) is odd, by symmetry we can assume that the initial condition \( x_0 \) is positive. Now \( \dot{x} = f(x) \) is always positive and \( x = x(t) \) is increasing on \( t \). If we write the solution as \( t = t(x) \) (we can do this because \( x \) is increasing), then the original differential equation is equivalent to
\[
t'(x) = 1/x'(t) = 1/(rx + x^3),
\]
with the initial condition \( t(x_0) = 0 \). Therefore
\[
t(x) = \int_{x_0}^{x} \frac{1}{rx + x^3}
\]
is always finite.

2.6.2 (Page 41) Suppose \( x(t) = x(t+T) = a \) for some \( T > 0 \). Then
\[
\int_{t}^{t+T} f(x) \frac{dx}{dt} dt = \int_{x(t)}^{x(t+T)} x(t+T)f(x) dx = \int_{a}^{a} f(x) dx = 0.
\]
However,
\[
\int_{t}^{t+T} f(x) \frac{dx}{dt} dt = \int_{t}^{t+T} \left( \frac{dx}{dt} \right)^2 dt
\]
Since the integrand is nonnegative and (as we have shown) the integral is 0, in fact we must have \( dx/dt = 0 \) from \( t \) to \( t + T \). Thus \( x(\cdot) \) is constant on this interval.

2.7.5 (Page 42)

The potential is \( V = \int \sinh(x) dx = \cosh(x) = (e^x + e^{-x})/2 \), which has a minimum at \( x = 0 \). Furthermore, \( V(x) \) increases as \( |x| \) increases (i.e., as \( x \) moves away from 0 in either direction.) Thus the origin is the only equilibrium point, and it is stable.

2.7.6 (Page 42)

Similar to the above \( V(x) = -rx - \frac{1}{2}x^2 + \frac{1}{4}x^4 \). Graphs of \( V(x) \) for some \( r \) values are shown in the figure. The equilibrium points are the local minima.