Math 120: Solutions to Sample Midterm Problems

1. (a) Let $Z$ be the kernel. If $x, y \in Z$, then $f(x) = f(y) = 1$, so $f(xy) = f(x)f(y) = (1)(1) = 1$, so $xy \in Z$. Also, if $x \in Z$, then $f(x) = 1$, so $f(x^{-1}) = f(x)^{-1} = 1^{-1} = 1$, so $x^{-1} \in Z$. Finally, $f(1) = 1$, so $1 \in Z$. (b) Suppose $x \in Z$ and $g \in G$. Then

$$f(gxg^{-1}) = f(g)f(x)f(g^{-1}) = f(g)1f(g^{-1}) = f(gg^{-1}) = f(1) = 1$$

so $gxg^{-1} \in Z$. Since this holds for every $x \in Z$ and $g \in G$, $Z$ is a normal subgroup of $G$.

2. This is a perfect shuffle of a 6 card deck, so $F = (1\ 2\ 4)(3\ 6\ 5)$ (as in a hw 3 problem). The order is the least common multiple of 3 and 3, namely 3.

3. This was in a recent hw assignment.

4. By Lagrange’s theorem, the only subgroups other than $\{1\}$ and $S_6$ itself each have order 2 or 3 and are therefore cyclic (generated by a single element). Of course 1 generates $\{1\}$, $r$ and $r^2$ both generated $\langle r \rangle = \{1, r, r^2\}$, and finally $s, rs$, and $r^2s$ generate cyclic groups of order 2. Thus the subgroups are $\langle 1 \rangle, \langle r \rangle, \langle s \rangle, \langle rs \rangle, \langle r^2s \rangle$, and $S_6$ itself.

5. By Lagrange’s theorem, the non-identity elements each have order $p$ or $p^2$. If any element $x$ has order $p^2$, then $\langle x \rangle = G$, so $G$ is cyclic. If not, then all non-identity elements have order $p$.

6. First $1s = s1$ for all $s \in S$, so $1 \in H$. If $x$ and $y$ are in $H$, then (for every $s \in S$)

$$(xy)s = xys = xsy = sxy = s(xy),$$

so $xy \in H$. Also, for every $s \in S$, $xs = sx$ so, multiplying on the left and on the right by $x^{-1}$, we see that $sx^{-1} = -x^{-1}s$. Thus $x^{-1} \in H$.

7. The intersection $A \cap B$ is a subgroup, so by Lagrange its order divides both $|A| = 15$ and $|B| = 21$. Thus $|A \cap B|$ is either 1 or 3. On the other hand

$$|AB| = \frac{|A||B|}{|A \cap B|} = \frac{15 \cdot 21}{|A \cap B|} = \frac{3 \cdot 105}{|A \cap B|}.$$  

This is 315 if $|A \cap B| = 1$ and 105 if $|A \cap B| = 3$. The former is impossible since $AB$ can’t have more elements than $G$. (It’s a subset of $G$.) Thus $|A \cap B| = 3$, and $|AB| = 105$. Thus $AB$ “fills up” all of $G$ (i.e., $G = AB$), which means every element of $G$ can be written as an element of $A$ times an element of $B$.

8. By Lagrange, $|A \cap B|$ must divide $3 (=|A| = |B|)$, so $|A \cap B|$ is 1 or 3. It can’t be 3, since then $A \cap B$ would fill up all of $A$ and of $B$ so that $A = A \cap B = B$, a contradiction. Thus $|A \cap B| = 1$. Then

$$|AB| = \frac{|A||B|}{|A \cap B|} = \frac{3 \cdot 3}{1} = 9.$$  

Since 9 does not divide $|G| = 30$, $AB$ cannot be a subgroup of $G$. Thus $AB \neq BA$ (see proposition 14, page 95.)