(Note: this way is harder!) In the invariant factor decomposition, \( n_1 \) must be divisible by each of the prime factors of \(|G|\) (see page 161). Thus \( n_1 \) is divisible by \( p \) and by \( q \), and therefore by \( pq \). Since \(|G|\) is divisible by \( n_1 \), this means \( n_1 = p^2q \) or \( n_1 = pq \). But \( n_1 < |G| \) since we are told that \( G \) is not cyclic. Thus \( n_1 = pq \), and therefore \( n_2 = p \). So \( G \cong \mathbb{Z}_{pq} \times \mathbb{Z}_p \).

Let us write \( \mathbb{Z}_{pq} \) and \( \mathbb{Z}_p \) additively. Now the order of \((a, b) \in G\) is

\[
\text{lcm}(|a|, |b|).
\]

The order of an element \( \bar{a} \in \mathbb{Z}_{pq} \) is \( pq/(a, pq) \). We may take \( 0 \leq a < pq \). If \( a \) is a multiple of \( p \) but not \( q \), then \( (a, pq) = p \), so \( |\bar{a}| = q \). There are \( q-1 \) such \( a \)'s, namely \( p, 2p, 3p, \ldots, (q-1)p \). Likewise if \( a \) is a multiple of \( q \) but not \( p \), then \( |\bar{a}| = p \). There are \( p-1 \) such \( a \)'s, namely \( q, 2q, \ldots, (p-1)q \). If \( a \) is a multiple of \( p \) and \( q \), then \( a = 0 \) and \( |\bar{a}| = 1 \). The other \( pq - (p-1) - (q-1) - 1 = pq - p - q + 1 = (p-1)(q-1) \) elements have order \( pq \).

Now \( \mathbb{Z}_p \) has one element (the identity) of order 1 and \( p-1 \) elements of order \( p \). Thus we have the following table:

<table>
<thead>
<tr>
<th></th>
<th>1 : 1</th>
<th>( p : q-1 )</th>
<th>( q : p-1 )</th>
<th>( pq : (p-1)(q-1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 : 1</td>
<td>1 : 1</td>
<td>( p : q-1 )</td>
<td>( q : p-1 )</td>
<td>( pq : (p-1)(q-1) )</td>
</tr>
<tr>
<td>( p : p-1 )</td>
<td>( p : p-1 )</td>
<td>( p : (p-1)(q-1) )</td>
<td>( pq : (p-1)^2 )</td>
<td>( pq : (p-1)^2(q-1) )</td>
</tr>
</tbody>
</table>

Order of element: number of elements with that order

The numbers across the top refer to elements \( \bar{a} \) of \( \mathbb{Z}_{pq} \). The numbers down the left side refer to elements \( \bar{b} \) of \( \mathbb{Z}_p \). The numbers in the table refer to the corresponding elements \((\bar{a}, \bar{b})\) of \( G \cong \mathbb{Z}_{pq} \times \mathbb{Z}_p \).

For example, looking at row 2 and column 3 of the table, we see that there are \( p-1 \) elements \( \bar{a} \) of order \( q \) in \( \mathbb{Z}_{pq} \), \( p-1 \) elements \( \bar{b} \) of order \( p \) in \( \mathbb{Z}_p \), and from these we get \((p-1)^2\) elements \((\bar{a}, \bar{b})\) of order \( pq \) in \( G \).

Altogether, \( G \) has:

- 1 element of order 1;
- \( p-1 \) elements of order \( q \);
- \((p-1)+(q-1)+(p-1)(q-1)=pq-1\) elements of order \( p \); and
- \((p-1)(q-1)+(p-1)^2(q-1)+(p-1)^2=p^2q-pq-p+1\) elements of order \( pq \).

In particular, the largest order is \( pq \).