§3.5 15: Let $H = \langle x \rangle$ and $K = \langle y \rangle$. Then $H = \{1, x, x^{-1}\}$, and since $y$ is not 1, $x$, or $x^{-1}$, $y \notin H$. Therefore $H \cap K$ is a subgroup of $K$, not equal to $K$, and so $|H \cap K| = 1$, and then $|HK| = |H||K|/|H \cap K| = 9$. Since $x, y$ are 3-cycles, they are both even. Therefore $\langle x, y \rangle$ is a subgroup of $A_4$ of order at least 9. But $A_4$ has order 12, and so $\langle x, y \rangle = A_4$.

§4.2 7:

(1) This is Cayley’s Theorem applied to $Q_8$.

(2) Suppose $n \leq 7$ and $\phi : Q_8 \to S_n$ is a homomorphism. This gives an action of $Q_8$ on $\{1, 2, \ldots, n\}$. For every $a$, since the orbit of $a$ has at most $n$ elements and $n < 8$, the stabilizer of $a$ in $Q_8$ must be nontrivial. But every nontrivial subgroup of $Q_8$ contains $-1$, so $-1$ stabilizes $a$. Since this holds for every $a$, $-1$ stabilizes everything, so $\phi(-1) = 1$. Therefore $\phi$ cannot be injective, so $Q_8$ cannot be isomorphic to a subgroup of $S_n$ with $n \leq 7$.

8: Consider the left cosets, $G/H$. $G$ acts on these cosets by left multiplication, which gives us a homomorphism of $G$ in $S_n$. By, theorem 3 (p. 121), the kernel $K$ of this action is the largest normal subgroup of $G$ contained in $H$. $K$ is the subgroup we are looking for. Notice $G \setminus K \subseteq S_n$, hence $|G : K| \leq |S_n| = n!$ as desired.

11: Consider the action of $H = \langle x \rangle$ on $G$ by left multiplication. Suppose $x \neq 1$. Note $x$ does not fix any element of $G$; that is, $xa \neq a$ for any $a \in G$. Thus stabilizer subgroup $H_a$ is $\{1\}$, so the orbit of $a$ has $n - 1$ elements (namely $a, xa, x^2a, \ldots, x^{n-1}a$). Thus each orbit has size $n$. Thus $|G|$ is the number $m$ of orbits times $n$. Such a permutation can be represented as the product of $(n-1)m$ transpositions, so it is odd if and only if $n - 1$ and $m$ are both odd, i.e. if and only if $n = |x|$ is even and $m = |G|/|x|$ is odd.

§4.3 5: We are given $|G : Z(G)| = n$. Notice the centralizer of any element must contain $Z(G)$, hence $|Z(g)| \leq |C_G(g_i)|$. Let $K_i$ be a conjugacy class. Now $|K_i| = |G : C_G(g_i)| \leq |G : Z(G)| = n$ as desired.

13: Suppose $G$ has finite order $n$ and has exactly two conjugacy classes. One of the conjugacy classes is $\{1\}$, so the other must be $G - \{1\}$. But the order of each conjugacy class divides $|G|$, so $n - 1$ divides
$n$ and therefore $n = 2$. Every group of order 2 is cyclic, so $G$ must be cyclic of order 2. Conversely, if $G$ is cyclic of order 2 then $G$ has exactly two conjugacy classes.

26: For $a \in A$, let $G_a$ denote the stabilizer of $a$, i.e. the set of $g \in G$ such that $g \cdot a = a$. We want an element of $G$ that does not stabilize any element of $A$. That is, we want a $g \in G$ that is not in any of the $G_a$’s. Now for each $a \in A$,

$$|A| = |\text{orbit}(a)| = |G|/|G_a|,$$

so $|G_a| = |G|/|A|$. Thus there are $|G|/|A| - 1$ nonidentity elements in $G_a$. Let $G^*_a$ denote the set of nonidentity elements of $G_a$. Then

$$|\bigcup_{a \in A} G_a| \leq |A|(|G|/|A| - 1) = |G| - |A|.$$

Thus $\bigcup_{a \in A} G_a$ has at most $|G| - |A| + 1 < |G|$ elements. So there must be at least one $g \in G$ that is not in any of the $G_a$’s.

30: If $G$ has odd order then every conjugacy class has odd order. Suppose $C$ is a conjugacy class and $x, x^{-1} \in C$. Then for every $y \in C$, say $y = gxg^{-1}$, we have $y^{-1} = gx^{-1}g^{-1}$ so $y^{-1} \in C$ as well. Therefore the sets $\{y, y^{-1}\}$ for $y \in C$ give a partition of $C$. But $C$ has odd order, so for at least one $y \in C$, we must have $|\{y, y^{-1}\}| = 1$, i.e., $y = y^{-1}$ and $y^2 = 1$. But $G$ has odd order, so we conclude that $y = 1$, $C = \{1\}$, and so $x = 1$.

§4.4 6: Let $H$ be a characteristic subgroup of a group $G$. Then $H$ is fixed by every automorphism of $G$. Since conjugation is an automorphism of $G$, then we must have $gHg^{-1} = H$ for every element $g \in G$, proving $H$ is normal. For the second part of the problem, consider the vector space $\mathbb{R}^2$. Every subgroup is normal since $\mathbb{R}^2$ is an abelian group under addition. Consider any one dimensional subspace, which is a subgroup. Rotation by 45 degrees is an automorphism of $\mathbb{R}^2$, but does not fix any one dimensional subspace.

8: (a) $K$ is normal $G$ means that conjugation by $g \in G$ fixes $K$. Recall that conjugation is an automorphism, hence since $H$ is characteristic in $K$, conjugation also fixes $H$, proving that $H$ is normal.

(b) $K$ characteristic in $G$ means that any automorphism of $G$ fixes $K$. Since $H$ is characteristic in $K$, this same automorphism also fixes $H$, proving $H$ is characteristic in $G$.

(c) Note that in any such example, $H$ has a conjugate $H' \neq H$, and $H'$ must be in $K$ (since $K$ is characteristic and therefore normal in $G$). Thus we need a $K$ that contains two isomorphic normal subgroups that are conjugate in a larger group $G$. If $K$ is abelian, then its subgroups will automatically be normal.
With that in mind, here is an example. Let $K$ be all translations of the plane. As our $H$ we can take all horizontal translations, and as $H'$ all vertical translations. We need $G$ to contain an element $g$ such that $g(a$ horizontal translation)$g^{-1}$ must be a vertical translation. Rotation by 90 degrees (about any point) does that. Thus let $G$ be the group whose elements are: all translations of the plane, together with all rotations through multiples of 90 degrees. (One has to check that this set is closed under the composition operation.) So: $H$ is normal in $K$, but not normal in $G$. It only remains to show that $K$ is characteristic in $G$. It is characteristic, because it is the set of all elements of infinite order, together with the identity element.

(If we think of the point $(x, y)$ in the plane as the complex number $x + iy$, then the elements of $G$ are maps of the form

$$f_{n,b}(z) = i^n z + b,$$

where $n$ is an integer and $b$ is a complex number. Rotation by 90 degrees about point $p$ is given by $z \mapsto i(z - p) + p = iz + (1 - i)p = f_{1,1-i}(z)$. Then $K$ is the set of $f_{n,b}$ with $n = 0$, $H$ is the set of $f_{n,b}$ with $n = 0$ and $b$ real. With this description, it is easy to check algebraically that $G$ is closed under composition.)

Perhaps some of you thought up simpler examples...