PUTNAM PROBLEM SOLVING SEMINAR: SIXTEEN ANALYSIS PROBLEMS (AND SOLUTIONS) FROM THE PUTNAM

Here are sixteen problems on the Putnam that are fundamentally about analysis, with solutions. They are taken (with very crude editing — hopefully not too much is garbled) from the forthcoming book on the Putnam, written with Kiran Kedlaya and Bjorn Poonen. Suggested usage: look at a problem that catches your eye, ponder it for ten minutes, then read the solution, trying to pick up new ideas, techniques, and theorems.

1. Problems

1986B4. For a positive real number \( r \), let \( G(r) \) be the minimum value of \( |r - \sqrt{m^2 + 2n^2}| \) for all integers \( m \) and \( n \). Prove or disprove the assertion that \( \lim_{r \to \infty} G(r) \) exists and equals 0.

1987B4. Let \((x_1, y_1) = (0.8, 0.6)\) and let \(x_{n+1} = x_n \cos y_n - y_n \sin y_n\) and \(y_{n+1} = x_n \sin y_n + y_n \cos y_n\) for \( n = 1, 2, \ldots \). For each of \( \lim_{n \to \infty} x_n \) and \( \lim_{n \to \infty} y_n \), prove that the limit exists and find it or prove that the limit does not exist.

1988A3. Determine, with proof, the set of real numbers \( x \) for which

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} \csc \frac{1}{n} - 1 \right)^x
\]

converges.

1988B3. For every \( n \) in the set \( \mathbb{Z}^+ = \{1, 2, \ldots \} \) of positive integers, let \( r_n \) be the minimum value of \( |c - d\sqrt{3}| \) for all nonnegative integers \( c \) and \( d \) with \( c + d = n \). Find, with proof, the smallest positive real number \( g \) with \( r_n \leq g \) for all \( n \in \mathbb{Z}^+ \).

1988B4. Prove that if \( \sum_{n=1}^{\infty} a_n \) is a convergent series of positive real numbers, then so is \( \sum_{n=1}^{\infty} (a_n)^{n/(n+1)} \).

1990A2. Is \( \sqrt{2} \) the limit of a sequence of numbers of the form \( \sqrt{n} - \sqrt{m} \), \((n, m = 0, 1, 2, \ldots)\)?

1992A4. Let \( f \) be an infinitely differentiable real-valued function defined on the real numbers. If

\[
f\left( \frac{1}{n} \right) = \frac{n}{n^2 + 1}, \quad n = 1, 2, 3, \ldots ,
\]

compute the values of the derivatives \( f^{(k)}(0) \), \( k = 1, 2, 3, \ldots \).

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*Date: November 23, 2001.*
1994B3. Find the set of all real numbers $k$ with the following property: For any positive, differentiable function $f$ that satisfies $f'(x) > f(x)$ for all $x$, there is some number $N$ such that $f(x) > e^{kx}$ for all $x > N$.

1995A5. Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable $f$ which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

for some constants $a_{ij} > 0$. Suppose that for all $i$, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

1996B2. Show that for every positive integer $n$,

$$\left( \frac{2n-1}{e} \right)^{\frac{n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left( \frac{2n+1}{e} \right)^{\frac{n+1}{2}}.$$

1996B4. For any square matrix $A$, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$ 

Prove or disprove: there exists a $2 \times 2$ matrix $A$ with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

1997B2. Let $f$ be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real $x$. Prove that $|f(x)|$ is bounded.

1998A3. Let $f$ be a real function on the real line with continuous third derivative. Prove that there exists a point $a$ such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$ 

1992B2. Let $P(x)$ be a polynomial of degree $n$ such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have $n$ distinct roots. [The roots may be either real or complex.]

2000A4. Show that the improper integral

$$\lim_{B \to \infty} \int_{0}^{B} \sin(x) \sin(x^2) \, dx$$
converges.

2000B3. Let \( f(t) = \sum_{j=1}^{N} a_j \sin(2\pi j t) \), where each \( a_j \) is real and \( a_N \) is not equal to 0. Let \( N_k \) denote the number of zeros (including multiplicities) of \( \frac{d^k f}{dt^k} \). Prove that

\[
N_0 \leq N_1 \leq N_2 \leq \cdots \quad \text{and} \quad \lim_{k \to \infty} N_k = 2N.
\]

2. Solutions

1986B4. First,

\[
0 \leq |r - \sqrt{m^2 + 2n^2}| = \frac{|r^2 - m^2 - 2n^2|}{r + \sqrt{m^2 + 2n^2}} \leq \frac{|r^2 - m^2 - 2n^2|}{r},
\]

so it will suffice to bound the latter expression. Select the largest integer \( m \geq 0 \) such that \( r^2 - m^2 \geq 0 \). Then \( m^2 \leq r^2 < (m+1)^2 \), so \( m \leq r \) and \( r^2 - m^2 < 2m + 1 \). Next select the largest integer \( n \geq 0 \) such that \( r^2 - m^2 - 2n^2 \geq 0 \). Then \( 2n^2 \leq r^2 - m^2 < 2(n+1)^2 \). This implies \( n \leq \sqrt{(r^2 - m^2)/2} \) and

\[
|r^2 - m^2 - 2n^2| = r^2 - m^2 - 2n^2 < 2(2n+1) \leq 2 + 4\sqrt{(r^2 - m^2)/2} \leq 2 + \sqrt{2m + 1} \leq 2 + \sqrt{2r + 1}.
\]

Hence \( G(r) \leq (2 + \sqrt{2r + 1})/r \) and \( \lim_{r \to \infty} G(r) = 0 \).

1987B4. Answer: Both limits exist: \( \lim_{n \to \infty} x_n = -1 \) and \( \lim_{n \to \infty} y_n = 0 \).

Since \((0.8)^2 + (0.6)^2 = 1\), we have \((x_1, y_1) = (\cos \theta_1, \sin \theta_1)\) where \( \theta_1 = \cos^{-1}(0.8) \). If \((x_n, y_n) = (\cos \theta_n, \sin \theta_n)\) for some \( n \geq 1 \) and number \( \theta_n \), then by the trigonometric addition formulas, \((x_{n+1}, y_{n+1}) = (\cos(\theta_n + y_n), \sin(\theta_n + y_n))\). Hence by induction, \((x_n, y_n) = (\cos \theta_n, \sin \theta_n)\) for all \( n \geq 1 \), where \( \theta_2, \theta_3, \ldots \) are defined recursively by \( \theta_{n+1} = \theta_n + y_n \) for \( n \geq 1 \). Thus \( \theta_{n+1} = \theta_0 + n \).

For \( 0 < \theta < \pi \), \( \sin \theta > 0 \) and \( \sin \theta = \sin(\pi - \theta) < \pi - \theta \), so \( 0 < \theta + \sin \theta < \pi \). By induction, \( 0 < \theta_n < \pi \) for all \( n \geq 1 \). Also \( \theta_{n+1} = \theta_n + \sin \theta_n > \theta_n \), so the bounded sequence \( \theta_1, \theta_2, \ldots \) is also increasing, and hence has a limit \( L \in [0, \pi] \).

Since \( \sin t \) is a continuous function, taking the limit as \( n \to \infty \) in \( \theta_{n+1} = \theta_n + \sin \theta_n \) shows that \( L = L + \sin L \), so \( \sin L = 0 \). But \( L \in [0, \pi] \) and \( L \geq \theta_1 > 0 \), so \( L = \pi \). By continuity of \( \cos t \) and \( \sin t \), \( \lim_{n \to \infty} x_n = \cos L = \cos \pi = -1 \) and \( \lim_{n \to \infty} y_n = \sin L = \cos \pi = 0 \).

1988A3. Answer: The series converges if and only if \( x > 1/2 \).

\(^1\text{The proposers intended for } N_k \text{ to count only the zeros in the interval } [0, 1].\)
Define
\[ a_n = \frac{1}{n} \csc \frac{1}{n} - 1 = \frac{1}{n \sin \frac{1}{n}} - 1. \]
Taking \( t = 1/n \) in the inequality \( 0 < \sin t < t \) for \( t \in (0, \pi) \), we obtain \( a_n > 0 \), so each term \( a_n^2 \) of the series is defined for any real \( x \). Using \( \sin t = t - t^3/3! + O(t^5) \) as \( t \to 0 \), we have, as \( n \to \infty \),
\[
a_n = \frac{1}{n \left( \frac{1}{n} - \frac{1}{6n^2} + O \left( \frac{1}{n^4} \right) \right)} - 1 = \frac{1}{1 - \frac{1}{6n^2} + O \left( \frac{1}{n^4} \right)} - 1 = \frac{1}{6n^2} + O \left( \frac{1}{n^4} \right).
\]
In particular, if \( b_n = 1/n^2 \), then \( a_n^2/b_n^2 \) has a finite limit as \( n \to \infty \), so by the Limit Comparison Test (Spivak Chapter 22 Theorem 2), \( \sum_{n=1}^\infty a_n^2 \) converges if and only if \( \sum_{n=1}^\infty b_n^2 = \sum_{n=1}^\infty n^{-2x} \) converges, which by the Integral Comparison Test (Spivak Chapter 22 Theorem 4) holds if and only if \( 2x > 1 \), i.e., \( x > 1/2 \).

**Very Useful Aside: Big-O and little-o notation.** Recall that \( O(g(n)) \) is a stand-in for a function \( f(n) \) for which there exists a constant \( C \) such that \( |f(n)| \leq C|g(n)| \) for all sufficiently large \( n \). (This does not necessarily imply that \( \lim_{n \to \infty} f(n)/g(n) \) exists.) Similarly “\( f(t) = O(g(t)) \) as \( t \to 0^+ \)” means that there exists a constant \( C \) such that \( |f(t)| \leq C|g(t)| \) for sufficiently small nonzero \( t \).

On the other hand, \( o(g(n)) \) is a stand-in for a function \( f(n) \) such that
\[
\lim_{n \to \infty} f(n)/g(n) = 0.
\]
One can similarly define “\( f(t) = o(g(t)) \) as \( t \to 0^+ \)”.

**1988B3. Answer:** The smallest such \( g \) is \((1 + \sqrt{3})/2 \).

Let \( g = (1 + \sqrt{3})/2 \). For each fixed \( n \), the sequence \( (n-1) - \sqrt{3}, (n-2) - 2\sqrt{3}, \ldots, -n\sqrt{3} \) is an arithmetic sequence with common difference \(-2g\) and with terms on both sides of 0, so there exists a unique term \( x_n \) in it with \(-g \leq x_n < g\). Then \( r_n = |x_n| \leq g \).

For \( x \in \mathbb{R} \), let “\((x \mod 1)\)” denote \( x - |x| \in [0, 1) \). Since \( \sqrt{3} \) is irrational, \( \{(-d\sqrt{3}) \mod 1 : d \in \mathbb{Z}^+\} \) is dense in \([0, 1) \). (See the remark below.) Hence for any \( \varepsilon > 0 \), we can find a positive integer \( d \) such that \( ((-d\sqrt{3}) \mod 1) \in (g-1-\varepsilon, g-1) \). Then \( c - d\sqrt{3} \in (g-\epsilon, g) \) for some integer \( c \geq 0 \). Let \( n = c + d \). Then \( r_n = x_n = c - d\sqrt{3} > g - \epsilon \) by the uniqueness of \( x_n \) above. Thus \( g \) cannot be lowered.

**1988B4. If** \( a_n \geq 1/2^{n+1} \), then \( a_n^{n/(n+1)} = a_n/a_n^{1/(n+1)} \leq 2a_n \). If \( a_n \leq 1/2^{n+1} \), then \( a_n^{n/(n+1)} \leq 1/2^n \). Hence \( a_n^{n/(n+1)} \leq 2a_n + 1/2^n \). But \( \sum_{n=1}^\infty (2a_n + 1/2^n) \) converges, so \( \sum_{n=1}^\infty a_n^{n/(n+1)} \) converges by the Comparison Test (Spivak Chapter 22 Theorem 1).
1990A2. Answer: Yes. In fact, every real number is a limit of numbers of the form $\sqrt[n]{n} - \sqrt[n]{m}$.

Solution 1. By the binomial expansion,

$$\sqrt{n+1} - \sqrt{n} = n^{1/3} \left( 1 + \frac{1}{n} \right)^{1/3} - n^{1/3} \left( 1 + O \left( \frac{1}{n} \right) \right) - n^{1/3} = O(n^{-2/3})$$

so $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$. (Alternatively, one could use

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{(n+1)^2 + \sqrt{n(n+1)} + \sqrt{n^2}}} = O(n^{-2/3})$$

or

$$\sqrt{n+1} - \sqrt{n} = \int_n^{n+1} \frac{d}{dx} \left( x^{1/3} \right) \, dx = \frac{1}{3} \int_n^{n+1} x^{-2/3} \, dx = O(n^{-2/3}).$$

It remains to show that for any increasing sequence $\{a_n\}$ with $a_n \to \infty$ and $a_{n+1} - a_n \to 0$, the set $S = \{a_n - a_m : m, n \geq 1\}$ is dense in $\mathbb{R}$. Given $r > 0$ and $\epsilon > 0$, fix $m$ such that $a_{M+1} - a_M < \epsilon$ for all $M \geq m$. If $n$ is the smallest integer with $a_n \geq a_m + r$, then $a_n < a_m + r + \epsilon$, so $a_n - a_m$ is within $\epsilon$ of $r$. This shows that $S$ is dense in $[0, \infty)$, and by symmetry $S$ is dense also in $(-\infty, 0]$.

Solution 2. Fix $r \in \mathbb{R}$ and $\epsilon > 0$. Then for sufficiently large positive integers $n$,

$$(n + r)^3 - (n + r - \epsilon)^3 = 3n^2 \epsilon + O(n) > 1,$$

so $(n + r - \epsilon)^3 \leq [(n + r)^3] \leq (n + r)^3$, and $\sqrt[3]{[(n + r)^3]}$ is within $\epsilon$ of $n + r$. Hence

$$\lim_{n \to \infty} \sqrt[3]{[(n + r)^3]} - \sqrt[3]{n} = r.$$

Solution 3. As in Solution 1, $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$, so the set $S = \{\sqrt{n} - \sqrt{m}\}$ contains arbitrarily small positive numbers. Also $S$ is closed under multiplication by positive integers $k$ since $k(\sqrt{n} - \sqrt{m}) = \sqrt{k^2n} - \sqrt{k^2m}$. Any set with the preceding two properties is dense in $[0, \infty)$, because any finite open interval $(a, b)$ in $[0, \infty)$ contains a multiple of any element of $S \cap (0, b-a)$. By symmetry $S$ is dense in $(-\infty, 0]$ too.

1992A4. Answer: We have

$$f^{(k)}(0) = \begin{cases} (-1)^{k/2} k! & \text{if } k \text{ is even;} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Let $g(x) = 1/(1 + x^2)$ and $h(x) = f(x) - g(x)$. The value of $g^{(k)}(0)$ is $k!$ times the coefficient of $x^k$ in the Taylor series $1/(1 + x^2) = \sum_{m=0}^{\infty} (-1)^m x^{2m}$, and the value of $h^{(k)}(0)$ is zero by the lemma below (which arguably is the main content of this problem). Thus

$$f^{(k)}(0) = g^{(k)}(0) + h^{(k)}(0) = \begin{cases} (-1)^{k/2} k! & \text{if } k \text{ is even;} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$
**Lemma.** Suppose $h$ is an infinitely differentiable real-valued function defined on the real numbers such that $h(1/n) = 0$ for $n = 1, 2, 3, \ldots$. Then $h^{(k)}(0) = 0$ for all nonnegative integers $k$.

This lemma can be proved in many ways (and is a special case of a more general result stating that if $h : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable, if $k \geq 0$, and if $a \in \mathbb{R}$, then $h^{(k)}(a)$ is determined by the values of $h$ on any sequence of distinct real numbers tending to $a$).

**Proof 1 (Rolle’s Theorem).** Since $h(x) = 0$ for a sequence of values of $x$ strictly decreasing to 0, $h(0) = 0$. By Rolle’s Theorem, $h'(x)$ has zeros between the zeros of $h(x)$; hence $h'(x) = 0$ for a sequence strictly decreasing to 0, so $h'(0) = 0$. Repeating this argument inductively, with $h^{(n)}(x)$ playing the role of $h(x)$, proves the lemma.

**Proof 2 (Taylor’s Formula).** We prove that $h^{(n)}(0) = 0$ by induction. The $n = 0$ case follows as in the previous proof, by continuity. Now assume that $n > 0$, and $h^{(k)}(0) = 0$ is known for $k < n$. Recall Taylor’s Formula (Lagrange’s form, see for example Apostol Section 7.7) which states that for any $x > 0$ and integer $n > 0$, there exists $\theta_n \in [0, x]$ such that

$$h(x) = h(0) + h'(0)x + \cdots + h^{(n-1)}(0)x^{n-1}/(n - 1)! + h^{(n)}(\theta_n)x^n/n!.$$  

By our inductive hypothesis,

$$h(0) = \cdots = h^{(n-1)}(0) = 0.$$  

Hence by taking $x = 1, 1/2, 1/3, \ldots$, we get $h^{(n)}(\theta_m) = 0$, where $0 \leq \theta_m \leq 1/m$. But $\lim_{m \to \infty} \theta_m = 0$, so by continuity $h^{(n)}(0) = 0$.

**Proof 3.** By continuity, $h(0) = 0$. Let $k$ be the smallest nonnegative integer such that $h^{(k)}(0) \neq 0$. We assume $h^{(k)}(0) > 0$; the same argument applies if $h^{(k)}(0) < 0$. Then there exists $\epsilon$ such that $h^{(k)}(x) > 0$ on $(0, \epsilon]$. Repeated integration shows that $h(x) > 0$ on $(0, \epsilon]$, a contradiction.

**Sketch of Proof 4 (explicit computation).** By definition,

$$h'(0) = \lim_{\epsilon \to 0} \frac{f(\epsilon) - f(0)}{\epsilon}.$$  

More generally, if $h$ is infinitely differentiable in a neighborhood of 0, then

$$h^{(k)}(0) = \lim_{\epsilon \to 0} \frac{\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} h^{(j)}(\epsilon)}{\epsilon^k}.$$  

(This can be proved by applying L’Hôpital’s Rule $k$ times to the expression on the right.) Then choose $\epsilon = 1/n$ where $n$ runs over the multiples of lcm$(1, \ldots, k)$, to obtain $h^{(k)}(x) = 0$.

**Remark.** The formula (1) holds under the weaker assumption that $h^{(k)}(0)$ exists. To prove this, apply L’Hôpital’s Rule $k - 1$ times, and then write the resulting
expression as a combination of limits of the form
\[ \lim_{\varepsilon \to 0} \frac{h^{(k-1)}(j\varepsilon) - h^{(k-1)}(0)}{j\varepsilon}, \]
each of which equals \( h^{(k)}(0) \), by definition.

Remark. Note that \( h(x) \) need not be the zero function! An infinitely differentiable function need not be represented by its Taylor series at a point, i.e., it need not be analytic. For example, consider
\[ h(x) = \begin{cases} e^{-1/x^2} \sin(\pi/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases} \]
It is infinitely differentiable for all \( x \), and all of its derivatives are 0 at \( x = 0 \). (It satisfies the hypotheses of the lemma!)

1994B3. Answer: The desired set is \((-\infty, 1)\).

Let \( h(x) = \ln f(x) - x \). Then the problem becomes that of determining for which \( k \) the following holds: if a real-valued function \( h(x) \) satisfies \( h'(x) > 0 \) for all \( x \), then there exists a number \( N \) such that \( h(x) > (k-1)x \) for all \( x > N \).

The function \(-e^{-x} \) is always negative, but has positive derivative, so no number \( k \geq 1 \) is in the set. (This corresponds to the function \( f(x) = e^{x-e^{-x}} \).) On the other hand any \( k < 1 \) is in the set: choose \( N \) such that \((k-1)N < h(0)\); then for \( x > N \), \( h(x) > h(0) > (k-1)N > (k-1)x \).

1995A5. Answer: Yes, the functions must be linearly dependent.

If \((v_1, \ldots, v_n)\) is a (complex) eigenvector of the matrix \((a_{ij})\) with (complex) eigenvalue \( \lambda \), then the function \( y = v_1 x_1 + \cdots + v_n x_n \) satisfies the differential equation
\[ \frac{dy}{dt} = \sum_{i,j} v_ia_{ij}x_j = \sum_j \lambda v_j x_j = \lambda y \]
Therefore \( y = ce^{\lambda t} \) for some \( c \in \mathbb{C} \).

Now the trace of the matrix \((a_{ij})\) is \( a_{11} + \cdots + a_{nn} \), which is positive. Since the trace is the sum of the eigenvalues of the matrix, there must be at least one eigenvalue \( \lambda = \alpha + i\beta \) with positive real part. Setting \( y = v_1 x_1 + \cdots + v_n x_n \), we have that
\[ y = ce^{\lambda t} = ce^{\alpha t}e^{i\beta t}. \]
We are given that \( |x_i| \to 0 \) as \( t \to \infty \) for \( i = 1, \ldots, n \), which implies that \( |y| \to 0 \) as well. On the other hand, for \( \alpha > 0 \), \( e^{\alpha t}e^{i\beta t} \) does not tend to 0 as \( t \to \infty \). Therefore we must have \( c = 0 \), and hence \( v_1 x_1 + \cdots + v_n x_n = 0 \). Since the \( x_i \) are linearly dependent over \( \mathbb{C} \), they are also linearly dependent over \( \mathbb{R} \), as desired.

Remark. We did not require the eigenvalue \( \lambda \) to be real. It turns out, however, that a matrix with nonnegative entries always has an eigenvector with nonnegative
entries, and its corresponding eigenvalue is then also nonnegative. Moreover, if the matrix has all positive entries, it has exactly one positive real eigenvector: this assertion is the Perron-Frobenius Theorem, which I discussed a few weeks ago.

**1996B2.** By estimating the area under the graph of \(\ln x\) using upper and lower rectangles of width 2 (see Figure 1), we get

\[
\int_1^{2n-1} \ln x \, dx < 2(\ln(3) + \cdots + \ln(2n - 1)) < \int_3^{2n+1} \ln x \, dx.
\]

Since \(\int \ln x \, dx = x \ln x - x + C\), exponentiating and taking square roots yields the middle two inequalities in

\[
\left( \frac{2n - 1}{e} \right)^{2n-1} e^{\frac{2n-1}{2}} e^{-n+1} < (2n - 1) \frac{2n-1}{e} e^{-n+1} \leq 1 \cdot 3 \cdots (2n - 1) \leq (2n + 1) \frac{2n+1}{e} \frac{e^{-n+1}}{3^{n/2}} < \left( \frac{2n + 1}{e} \right)^{2n+1},
\]

and the inequalities at the ends follow from \(1 < e < 3\).

![Figure 1](image.png)

**Figure 1.** Estimating \(\int \ln x \, dx\).

**Remark.** You may enjoy using this method to prove that

\[e(n/e)^n < n! < en(n/e)^n,\]

which is an elementary first approximation to Stirling’s formula.
Better estimates for \( n! \) can be obtained using the Euler-Maclaurin summation formula: for any fixed \( k > 0 \),
\[
\sum_{j=a}^{b} f(j) = \int_{a}^{b} f(t) \, dt + \frac{f(a) + f(b)}{2} + \sum_{i=1}^{k} \frac{B_{2i}}{(2i)!} \left( f^{(2i-1)}(b) - f^{(2i-1)}(a) \right) + R_k(a, b),
\]
where the Bernoulli numbers \( B_{2i} \) are given by the power series
\[
x \frac{e^{x} - 1}{e^x - 1} = x + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} x^{2i},
\]
(see Hardy and Wright, Section 17.2) and the error term \( R_k(a, b) \) is given by
\[
R_k(a, b) = \frac{-1}{(2k + 2)!} \int_{a}^{b} B_{2k+2}(t-[t]) f^{(2k+2)}(t) \, dt.
\]
Specifically, this formula can be used to obtain Stirling’s approximation to \( n! \)
\[
n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n,
\]
where the tilde indicates that the ratio of the two sides tends to 1 as \( n \to \infty \).

The Euler-Maclaurin formula has additional applications in numerical analysis, as well as in combinatorics.

1996B4. Answer: There does not exist such a matrix \( A \).

Solution 1. Over the complex numbers, if \( A \) has distinct eigenvalues, it is diagonalizable. Since \( \sin A \) is a convergent power series in \( A \), eigenvectors of \( A \) are also eigenvectors of \( \sin A \), so \( A \) having distinct eigenvalues would imply that \( \sin A \) is diagonalizable. Since \( \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix} \) is not diagonalizable, it can be \( \sin A \) only for a matrix \( A \) with equal eigenvalues. This matrix can be conjugated into the form
\[
\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}
\]
for some \( x \) and \( y \). Using the power series for \( \sin \), we compute
\[
\sin \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}^{2k+1}
= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!} \begin{pmatrix} 1 & y/x \\ 0 & 1 \end{pmatrix}^{2k+1}
= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!} \begin{pmatrix} 1 & (2k + 1)y/x \\ 0 & 1 \end{pmatrix}
= \left( \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!} \right) \begin{pmatrix} 0 & \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!} \\ \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!} & 0 \end{pmatrix}
= \begin{pmatrix} \sin x & y \cos x \\ 0 & \sin x \end{pmatrix}.
\]
Thus if $\sin x = 1$, then $\cos x = 0$ and $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ is the identity matrix. In other words, $\sin A$ cannot equal a matrix whose eigenvalues are 1 but which is not the identity matrix.

**Remark.** The computation of $\sin \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ can be simplified by taking $A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ and $B = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ in the identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

which holds when $A$ and $B$ commute.

**Solution 2.** Put $\cos A = \sum_{n=0}^{\infty} (-1)^n A^{2n}/n!$. The identity $\sin^2 x + \cos^2 x = 1$ implies the identity

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1}\right)^2 + \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right)^2 = 1$$

of formal power series. The series converge absolutely if we substitute $x = A$, so $\sin^2 A + \cos^2 A$ equals the identity matrix $I$. But

$$I - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 3992 \\ 0 & 0 \end{pmatrix}$$

cannot be the square of a matrix; such a matrix would have to be nilpotent, and the $k^{th}$ power of a $k \times k$ nilpotent matrix is always zero, by the Cayley-Hamilton Theorem.

1997B2. Multiplying both sides of the given equation by $2f(x)$, we have

$$2f(x)f'(x) + 2f'(x)f''(x) = -2xg(x)f'(x)^2.$$

The left side of the equation is the derivative of $f(x)^2 + f'(x)^2$, whereas the right side is nonnegative for $x < 0$ and nonpositive for $x > 0$. Thus $f(x)^2 + f'(x)^2$ increases to its maximum value at $x = 0$ and decreases thereafter. In particular, it is bounded, so $f(x)$ and $f'(x)$ are bounded.

**Remark.** This problem has a physical interpretation: $f(x)$ is the amplitude of an oscillator with time-dependent damping $xg(x)$. Since the damping is negative for $x < 0$ and positive for $x > 0$, the oscillator gains energy before time 0 and loses energy thereafter.

1998A3. If at least one of $f(a)$, $f'(a)$, $f''(a)$, or $f'''(a)$ vanishes at some point $a$, then we are done. Otherwise by the Intermediate Value Theorem, each of $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ is either strictly positive or strictly negative on the real line, and we only need to show that their product is positive for a single value of $x$. By replacing $f(x)$ by $-f(x)$ if necessary, we may assume $f''(x) > 0$; by replacing $f(x)$ by $f(-x)$ if necessary, we may assume $f'''(x) > 0$. Notice that these substitutions do not change the sign of $f(x)f'(x)f''(x)f'''(x)$. 

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Now \( f'''(x) > 0 \) implies that \( f''(x) \) is increasing. Thus for \( a > 0 \),

\[
f'(a) = f'(0) + \int_0^a f''(t) \, dt \geq f'(0) + a f''(0).
\]

In particular, \( f'(a) > 0 \) for large \( a \). Similarly, since \( f''(x) > 0 \), \( f(a) \) is positive for large \( a \). Therefore \( f(x)f'(x)f''(x)f'''(x) > 0 \) for sufficiently large \( x \).

Remark. More succinctly, \( f \) cannot both be positive and strictly concave-down everywhere, nor negative and strictly concave-up everywhere. So \( f(x)f'''(x) \) must be positive for some range of \( x \), as must be \( f'(x)f''(x) \) by the same reasoning applied to \( f' \) instead of \( f \).

1999B2. Suppose that \( P \) does not have \( n \) distinct zeros; then it has a zero of multiplicity \( k \geq 2 \), which we may assume without loss of generality is \( x = 0 \). Differentiating \( P \) term by term shows that the highest power of \( x \) dividing \( P'(x) \) is \( x^{k-2} \). But \( P(x) = Q(x)P'(x) \), so \( x^2 \) divides \( Q(x) \). Since \( Q \) is quadratic, \( Q(x) = Cx^2 \) for some constant \( C \). Comparing the leading coefficients of \( P(x) \) and \( Q(x)P'(x) \) yields \( C = \frac{1}{x^{k-2}} \).

Write \( P(x) = \sum_{j=0}^{n} a_j x^j \); equating coefficients in \( P(x) = Cx^2 P'(x) \) implies that \( a_j = C j(j-1) a_j \) for all \( j \). Hence \( a_j = 0 \) for \( j \leq n - 1 \), and \( P(x) = a_n x^n \), which has all identical zeroes.

2000A4. Solution 1. We may shift the lower limit to 1 without affecting convergence. That done, we use integration by parts:

\[
\int_1^B \sin x \sin x^2 \, dx = \int_1^B \frac{\sin x}{2x} \sin x^2 (2x \, dx)
\]

\[
= -\left[ \frac{\sin x}{2x} \cos x^2 \right]_1^B + \int_1^B \frac{\cos x}{2x} \left( \frac{\sin x}{2} - \frac{\sin x}{2x^2} \right) \cos x^2 \, dx.
\]

Now \( \frac{\sin x}{2x} \cos x^2 \) tends to 0 as \( x \to \infty \), and the integral of \( \frac{\sin x}{2x} \cos x^2 \) converges absolutely as \( B \to \infty \) by comparison to \( 1/x^2 \). It remains to consider

\[
\int_1^B \frac{\cos x}{2x} \cos x^2 \, dx = \int_1^B \frac{\cos x}{4x^2} \cos x^2 (2x \, dx)
\]

\[
= \left[ \frac{\cos x}{4x^2} \sin x^2 \right]_1^B - \int_1^B \frac{2 \cos x - x \sin x}{4x^3} \sin x^2 \, dx.
\]

Now \( \frac{\cos B}{4x^2} \sin B^2 \to 0 \) as \( B \to \infty \), and the final integral converges absolutely as \( B \to \infty \) by comparison to the integral of \( 1/x^2 \).

Solution 2. The addition formula for cosine implies that \( \sin x \sin x^2 = \frac{1}{2}[\cos(x^2 - x) - \cos(x^2 + x)] \). The substitution \( x = y + 1 \) transforms \( x^2 - x \) into \( y^2 + y \), so it suffices to show that \( \int_0^\infty \cos(x^2 + x) \, dx \) converges. Now substitute \( u = x^2 + x \); then \( x = -1/2 + \sqrt{u + 1}/4 \) and

\[
\int_0^\infty \cos(x^2 + x) \, dx = \int_0^\infty \frac{\cos u}{2\sqrt{u + 1}/4} \, du.
\]
The latter integrand is bounded, so we may replace the lower limit of integration with $\frac{\pi}{2}$ without affecting convergence. Moreover, the integrand tends to zero as $u \to \infty$, so the error introduced by replacing the upper limit of integration in
\[
\int_{\frac{\pi}{2}}^{B} \frac{\cos u}{2\sqrt{u + 1/4}} \, du
\]
by the nearest odd integer multiple of $\frac{\pi}{2}$ tends to zero as $B \to \infty$. Therefore (1) converges if and only if \( \sum_{n=1}^{\infty} a_n \) converges, where
\[
a_n = \int_{(n-\frac{1}{2})\pi}^{(n+\frac{1}{2})\pi} \frac{\cos u}{2\sqrt{u + 1/4}} \, du = (-1)^n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos t}{2\sqrt{t + n\pi + 1/4}} \, dt.
\]
The integrand in the last expression decreases to 0 uniformly as $n \to \infty$. Therefore $a_n$ alternate in sign, tend to 0 as $n \to \infty$ and satisfy $|a_n| \geq |a_{n+1}|$. By the alternating series test, \( \sum_{n=1}^{\infty} a_n \) converges, so the original integral converges.

2000B3. We first show $N_k \leq 2N$ for all $k \geq 0$. Write $f^{(k)}(t)$ for \( \frac{d^k}{dt^k} \). If we use the identity $\sin x = (e^{ix} - e^{-ix})/(2i)$ and set $z = e^{2\pi i t}$, then
\[
f^{(k)}(t) = \frac{1}{2i} \sum_{j=1}^{N} (2\pi i j)^k a_j (z^j - z^{-j})
\]
is rewritten as $z^{-N}$ times a polynomial of degree $2N$ in $z$. Hence as a function of $z$, it has at most $2N$ zeros. Therefore $f_k(t)$ has at most $2N$ zeros in $[0, 1]$; that is, $N_k \leq 2N$. In particular, $f^{(k)}$ has at most finitely many zeros in $[0, 1]$.

To prove $N_k \leq N_{k+1}$, we use Rolle’s Theorem, and the fact that at every zero of $f, f'$ has a zero of multiplicity one less. If $0 = t_1 < t_2 < \cdots < t_r < 1$ are the zeros of $f^{(k)}$ in $[0, 1]$, occurring with respective multiplicities $m_1, \ldots, m_r$, then $f^{(k+1)}$ has at least one zero in each of the open intervals $(t_1, t_2), (t_2, t_3), \ldots, (t_r-1, t_r), (t_r, t_1 + 1)$; we may translate the part $[1, t_1 + 1)$ of the last interval to $[0, t_1)$, on which $f^{(k+1)}$ takes the same values. This gives $r$ zeros of $f^{(k+1)}$. Adding to these the multiplicities $m_1 - 1, \ldots, m_r - 1$ at $t_1, \ldots, t_r$, we find that $f^{(k+1)}$ has at least $m_1 + \cdots + m_r = N_k$ zeros. Thus $N_{k+1} \geq N_k$.

To establish that $N_k \to 2N$, it suffices to prove $N_{4k} \geq 2N$ for sufficiently large $k$. This we do by making precise the assertion that
\[
f^{(4k)}(t) = \sum_{j=1}^{N} (2\pi j)^{4k} a_j \sin(2\pi j t)
\]
is dominated by the term with $j = N$ at each point $t = t_m = (2m + 1)/(4N)$ for $m = 0, 1, \ldots, 2N - 1$. At $t = t_m$, the $j = N$ term is
\[
(2\pi N)^{4k} a_N \sin(2\pi N t_m) = (-1)^m (2\pi N)^{4k} a_N.
\]
The absolute value of the sum of the other terms at $t = t_m$, divided by the $j = N$ term, is bounded by
\[
|a_1|^k \left( \frac{1}{N} \right)^{4k} + \cdots + |a_{N-1}| \left( \frac{N-1}{N} \right)^{4k} \frac{1}{N}.
\]
which tends to 0 as $k \to \infty$; in particular, this ratio is less than 1 for sufficiently large $k$. Then $f^{(4k)}(t_m)$ has the same sign as $(-1)^{m}(2\pi N)^{4k}a_N$. In particular, the sequence

$$f^{(4k)}(t_0), f^{(4k)}(t_1), \ldots, f^{(4k)}(t_{2N-1})$$

alternates in sign, when $k$ is sufficiently large. Between these points (again including a final “wraparound” interval) we find $2N$ sign changes of $f^{(4k)}$. By the Intermediate Value Theorem, this implies $N_{4k} \geq 2N$ for large $k$, and we are done.

Remark. A more analytic proof that $N_k \to 2N$ involves the observation that $(2\pi N)^{-4k} f^{(4k)}(x)$ and its derivative converge uniformly to $a_N \sin(2\pi N t)$ and $2\pi N a_N \cos(2\pi N t)$, respectively, on $[0, 1]$. That implies that for large $k$, we can divide $[0, 1]$ into intervals on which $f^{(4k)}(x)$ does not change sign, and intervals on which $f^{(4k+1)}(x)$ does not change sign while $f^{(4k)}(x)$ has a sign change.

This handout, and other useful things, can (soon) be found at

http://math.stanford.edu/~vakil/stanfordputnam.html