The Rules. These are way too many problems to consider. Just pick a few problems in one of the sections and play around with them. You are not allowed to try a problem that you already know how to solve.

Generating functions.

1. Suppose \( p(x) = (1 + x + x^2)^{2001} \) is expanded out as a huge degree 4002 polynomial

\[
p(x) = a_0 + a_1 x + \cdots + a_{4001} x^{4001} + a_{4002} x^{4002}.
\]

(a) Find the sum of the coefficients of \( p(x) \). (b) Find the sum of the even coefficients of \( p(x) \). (Hint: What is \( p(-1) \)?) (A challenge: Find the sum of every third coefficient — it turns out to be a power of 3.)

2. Suppose

\[
x = 0.12345\ldots = \sum_{i=1}^{\infty} \frac{i}{10^i}.
\]

(a) What is the thousandth digit of \( x \) after the decimal place? (b) Show that \( x \) is a rational number. Find it.

3. Show that

\[
0.0001 0016 0081 0256 \ldots = \sum_{i=1}^{\infty} \frac{i^4}{10^{4i}}
\]

is a rational number. (Tip: Don’t try to find it!)

4. \( 1/9899 = 0.0001000100020003000500080013 \ldots \) (As in the previous problem, the spaces were added to make the pattern clear.) Explain! Can you generalize it? For example, which rational number is \( 0.000 \ldots \)?

5. Notice that \( e^{ax}e^{bx} = e^{(a+b)x} \). Consider both sides as power series. Write down the coefficient of \( x^n \) on each side. What equality have you just proved? (Recall that \( e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!} \).)

6. For nonnegative integers \( n \) and \( k \), define \( Q(n, k) \) to be the coefficient of \( x^k \) in the expansion of \( (1 + x + x^2 + x^3)^n \). Prove that

\[
Q(n, k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k - 2j},
\]

Date: October 23, 2001.
where \((\binom{a}{b})\) is the standard binomial coefficient. (Reminder: For integers \(a\) and \(b\) with \(a \geq 0\), \((\binom{a}{b}) = \frac{a!}{b!(a-b)!}\) for \(0 \leq b \leq a\), with \((\binom{a}{b}) = 0\) otherwise.) Here’s a hint which may help: Note that \((1 + x + x^2 + x^3)\) factors.

**Analysis on the real line: Handy facts to know (for the Putnam, and more generally for a long and happy life).**


*Continuity.* The Intermediate Value Theorem. The Extreme Value Theorem. (More generally, a continuous function on a compact set attains its sup and inf.)

Descartes’ Rule of Signs: If \(p(x) = a_1x^{r_1} + a_2x^{r_2} + \cdots + a_kx^{r_k}\) is a polynomial with \(a_1 \in \mathbb{R}^+\) and \(r_1 > r_2 > \cdots > r_k\), then the number of positive real zeros of \(p(x)\) counted with multiplicity is the number of sign changes in the sequence \(a_1, a_2, \ldots, a_k\) minus a nonnegative even integer.

Big \(O\) and little \(o\) notation: \(O(g(n))\) is a stand-in for a function \(f(n)\) for which there exists a constant \(C\) such that \(|f(n)| \leq C|g(n)|\) for all sufficiently large \(n\). (This does not necessarily imply that \(\lim_{n \to \infty} f(n)/g(n) = 0\).) Similarly “\(f(t) = O(g(t))\) as \(t \to 0^+\)” means that there exists a constant \(C\) such that \(|f(t)| \leq C|g(t)|\) for sufficiently small nonzero \(t\). \(o(g(n))\) is a stand-in for a function \(f(n)\) such that \(\lim_{n \to \infty} f(n)/g(n) = 0\). One can similarly define “\(f(t) = o(g(t))\) as \(t \to 0^+\)”.

*Calculus.* Riemann Sums: if a function is Riemann-integrable, e.g. if it is continuous on a closed finite interval, then the integral is the limit of the Riemann sums.

Rolle’s Theorem: Let \([a, b]\) be a closed interval in \(\mathbb{R}\). Let \(f(t)\) be a function that is continuous on \([a, b]\) and differentiable on \((a, b)\), and suppose that \(f(a) = f(b)\). Then there exists \(c \in (a, b)\) such that \(f’(c) = 0\).

*Inequalities of integrals:* \(f \leq g\) means \(\int_a^b f \leq \int_a^b g\) if \(a \leq b\).

Taylor’s Formula with Remainder: if \(h\) has continuous \(n\)th derivatives, then for any \(x > 0\) and integer \(n > 0\), there exists \(\theta_n \in [0, x]\) such that \(h(x) = h(0) + h’(0)x + \cdots + h^{(n-1)}(0)x^{n-1}/(n-1)! + h^{(n)}(\theta_n)x^n/n!\).

Mean Value Theorem for integrals: If \(f\) is continuous on \([a, b]\), then for some \(c\) in \([a, b]\) we have \(\int_a^b f(x)dx = f(c)(b-a)\). For derivatives: If \(f\) is continuous on \([a, b]\) and has a derivative at each point of \((a, b)\), then there is a point \(c\) of \((a, b)\) for which \(f(b) - f(a) = f’(c)(b-a)\).

*Always good to know.* Ordinary differential equations

*Random other facts.*
Rouché’s Theorem: If $f$ and $g$ are analytic functions on an open set of $\mathbb{C}$ containing a closed disc, and if $|g(z) - f(z)| < |f(z)|$ everywhere on the boundary of the disc, then $f$ and $g$ have the same number of zeros inside the disc.

Euler-Maclaurin Summation Formula: for any fixed $k > 0$,

$$\sum_{j=a}^{b} f(j) = \int_{a}^{b} f(t) \, dt + \frac{f(a) + f(b)}{2} + \sum_{i=1}^{k} \frac{B_{2i}}{(2i)!} \left( f^{(2i-1)}(b) - f^{(2i-1)}(a) \right) + R_k(a, b),$$

where the Bernoulli numbers $B_{2i}$ are given by the power series

$$xe^x - 1 = 1 - x/2 + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} x^{2i},$$

and the error term $R_k(a, b)$ is given by

$$R_k(a, b) = -\frac{1}{(2k+2)!} \int_{a}^{b} B_{2k+2}(t - \lfloor t \rfloor) f^{(2k+2)}(t) \, dt.$$

Application 1: Sum of $k$th powers. Application 2: Stirling’s approximation to $n!$,

$$n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n,$$

where the tilde indicates that the ratio of the two sides tends to 1 as $n \to \infty$.

Problems I’ll discuss.

7. Let $f$ be an infinitely differentiable real-valued function defined on the real numbers. If

$$f \left( \frac{1}{n} \right) = \frac{n^2}{n^2 + 1}, \quad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \ldots$.

8. For any pair $(x, y)$ of real numbers, a sequence $(a_n(x, y))_{n \geq 0}$ is defined as follows:

$$a_0(x, y) = x,$$

$$a_{n+1}(x, y) = \frac{(a_n(x, y))^2 + y^2}{2}, \quad \text{for } n \geq 0.$$

Find the area of the region $\{(x, y) | (a_n(x, y))_{n \geq 0} \text{ converges} \}$.

9. Let $a$ and $b$ be positive numbers. Find the largest number $c$, in terms of $a$ and $b$, such that

$$a^x b^{1-x} \leq a \frac{\sinh ux}{\sinh u} + b \frac{\sinh u(1-x)}{\sinh u}$$

for all $u$ with $0 < |u| \leq c$ and for all $x$, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

Other problems.

10. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.
11. **Cauchy’s Lemma.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function such that \( f(x + y) = f(x) + f(y) \). Show that \( f(x) = cx \) for some \( c \in \mathbb{R} \).

12. **A first approximation to Stirling’s formula.** Prove that \( e(n/e)^n < n! < en(n/e)^n \). (Hint: Use Riemann sums on \( y = \ln x \).)

13. Is there an infinite sequence \( a_0, a_1, a_2, \ldots \) of nonzero real numbers such that for \( n = 1, 2, 3, \ldots \) the polynomial

\[
p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n
\]

has exactly \( n \) distinct real roots?

14. Let \( N \) be the positive integer with 1998 decimal digits, all of them 1; that is,

\[
N = 1111\cdots11.
\]

Find the thousandth digit after the decimal point of \( \sqrt{N} \).

*This handout, and other useful things, can (soon) be found at*

\[
http://math.stanford.edu/~vakil/stanfordputnam.html
\]