THE FIFTY-FOURTH WILLIAM LOWELL PUTNAM
MATHEMATICAL COMPETITION (DECEMBER 4, 1993)

A1. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find $c$ so that the areas of the two shaded regions are equal.

![Diagram](image)

A2. Let $(x_n)_{n \geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1 \text{ for } n = 1, 2, 3, \ldots.$$  

Prove there exists a real number $a$ such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

A3. Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$  

A4. Let $x_1, x_2, \ldots, x_{10}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{63}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some $x_i$’s equal to a sum of some $y_j$’s.

A5. Show that

$$\int_{-100}^{100} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{1/100}^{1/10} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{1/10}^{1/10} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx$$

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is a rational number.

**A6.** The infinite sequence of 2’s and 3’s

\[ 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, \ldots \]

has the property that, if one forms a second sequence that records the number of
3’s between successive 2’s, the result is identical to the given sequence. Show that
there exists a real number \( r \) such that, for any \( n \), the \( n^{th} \) term of the sequence is
2 if and only if \( n = 1 + [rm] \) for some nonnegative integer \( m \). (Note: \([x]\) denotes
the largest integer less than or equal to \( x \).)

**B1.** Find the smallest positive integer \( n \) such that for every integer \( m \), with
\( 0 < m < 1003 \), there exists an integer \( k \) for which

\[
\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.
\]

**B2.** Consider the following game played with a deck of \( 2n \) cards numbered
from 1 to \( 2n \). The deck is randomly shuffled and \( n \) cards are dealt to each of two
players, \( A \) and \( B \). Beginning with \( A \), the players take turns discarding one of their
remaining cards and announcing its number. The game ends as soon as the sum
of the numbers on the discarded cards is divisible by \( 2n + 1 \). The last person to
discard wins the game. Assuming optimal strategy by both \( A \) and \( B \), what is the
probability that \( A \) wins?

**B3.** Two real numbers \( x \) and \( y \) are chosen at random in the interval (0,1) with
respect to the uniform distribution. What is the probability that the closest integer
to \( x/y \) is even? Express the answer in the form \( r + sp \), where \( r \) and \( s \) are rational
numbers.

**B4.** The function \( K(x, y) \) is positive and continuous for \( 0 \leq x \leq 1, 0 \leq y \leq 1 \),
and the functions \( f(x) \) and \( g(x) \) are positive and continuous for \( 0 \leq x \leq 1 \). Suppose
that for all \( x, 0 \leq x \leq 1, \)

\[
\int_0^1 f(y)K(x, y) \, dy = g(x) \quad \text{and} \quad \int_0^1 g(y)K(x, y) \, dy = f(x).
\]

Show that \( f(x) = g(x) \) for \( 0 \leq x \leq 1 \).

**B5.** Show there do not exist four points in the Euclidean plane such that the
pairwise distances between the points are all odd integers.

**B6.** Let \( S \) be a set of three, not necessarily distinct, positive integers. Show
that one can transform \( S \) into a set containing 0 by a finite number of applications
of the following rule: Select two of the three integers, say \( x \) and \( y \), where \( x \leq y \) and
replace them with \( 2x \) and \( y - x \).

This handout, and other useful things, can (soon) be found at

http://math.stanford.edu/~vakil/stanfordputnam.html