The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.


Problem of the Week: volume of $n$-dimensional spheres.

Let $S_n(R)$ be the “$n$-dimensional sphere of radius $R$”. For example, $S_3(R)$ is the sphere of radius $R$; $S_2(R)$ is the (interior of the) circle of radius $R$; $S_1(R)$ is a line segment of length $2R$ (why?).

1. Make a table of values of the “volume” $V_n(R)$ and “surface area” $A_n(R)$ of $S_n(R)$ for $n = 2, 3, 4, 5$. (This is an example of “generalizing downward”, and will require some creative thinking.)

\[
\begin{array}{cccccc}
\text{ } & 0 & 1 & 2 & 3 & 4 & 5 \\
V_n(R) & \text{“volume” of } S_n(R) \\
A_n(R) & \text{“surface area” of } S_n(R)
\end{array}
\]

2. Why is $S_n(R)$ a constant multiple of $R^n$? Why is $A_n(R)$ a constant multiple of $R^{n-1}$?

3. Why is $A_n(R) = \frac{d}{dr} S_n(R)$?

In fact,

\[
S_n(R) = \frac{\pi^{n/2}}{(n/2)!} r^n.
\]

4. “But wait!” you exclaim — “we don’t know the meaning of $(n/2)!$ when $n$ is odd!” So using the table, define $(1/2)!$. Then define $n! = n(n-1)!$ even when $n$ is a half-integer.

Date: Tuesday, November 19, 2002.
Check your answer by verifying that the resulting formula for the volume of the 3-sphere still works.

5. Prove equation (1) by induction as follows. Prove it for $n = 1$ and 2. Then prove it for $n$ assuming it holds for $n - 2$, by showing that

$$
\text{vol } S_n(R) = \iiint_{x \in \text{n-sphere of radius } R} 1
= \iint_{y \in \text{circle of radius } R} \left( \iiint_{\tilde{x} \in (n-2)\text{-sphere of radius } \sqrt{R^2 - |y|^2}} 1 \right)
$$

and computing the nested integrals on the right. (At some point, polar coordinates may help.)

6. (This is easier than many of the earlier ones.) There is a less ad hoc definition of $n!$ when $n$ isn’t an integer. If $s$ is a non-negative real number, define $g(s) = \int_0^\infty x^s e^{-x} dx$. Prove that (i) $g(0) = 1$, (ii) $g(s) = sg(s - 1)$ if $s \geq 1$, assuming that $g(s - 1)$ exists. Be careful with convergence! Hence $g(s) = s!$ when $s$ is an integer.

This function, shifted by one, is called the “gamma function”. There is a neat argument that $g(1/2)$ is the value you found in problem 5; thus $g(s) = s!$ even when $s$ is a half-integer.

Putnam Problems.

1987A3. For all real $x$, the real-valued function $y = f(x)$ satisfies

$$
y'' - 2y' + y = 2e^x.
$$

(a) If $f(x) > 0$ for all real $x$, must $f'(x) > 0$ for all real $x$? Explain.
(b) If $f'(x) > 0$ for all real $x$, must $f(x) > 0$ for all real $x$? Explain.

1987B1. Evaluate

$$
\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x) + \sqrt{\ln(x+3)}}} dx.
$$

1995A2. For what pairs $(a, b)$ of positive real numbers does the improper integral

$$
\int_b^\infty \left( \sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx
$$

converge?

1991B2. Suppose $f$ and $g$ are nonconstant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers $x$ and $y$,

$$
f(x + y) = f(x)f(y) - g(x)g(y),
g(x + y) = f(x)g(y) + g(x)f(y).
$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all $x$. 
1997B2. Let $f$ be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -x g(x) f'(x),$$

where $g(x) \geq 0$ for all real $x$. Prove that $|f(x)|$ is bounded.

1998A3. Let $f$ be a real function on the real line with continuous third derivative. Prove that there exists a point $a$ such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

1991A5. Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2} \, dx$$

for $0 \leq y \leq 1$.

1993A5. Show that

$$\int_{-10}^{-\frac{1}{10}} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \, dx + \int_{\frac{1}{10}}^{\frac{1}{10}} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \, dx + \int_{\frac{11}{10}}^{\frac{11}{10}} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \, dx$$

is a rational number.

1994B3. Find the set of all real numbers $k$ with the following property: For any positive, differentiable function $f$ that satisfies $f'(x) > f(x)$ for all $x$, there is some number $N$ such that $f(x) > e^{kx}$ for all $x > N$.

1997A3. Evaluate

$$\int_0^\infty \left( x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left( 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) \, dx.$$

This handout can (soon) be found at

http://math.stanford.edu/~vakil/stanfordputnam/

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