The Rules. These are way too many problems to consider. Just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.


Problem of the Week: volume of $n$-dimensional spheres.

Let $S_n(R)$ be the “$n$-dimensional sphere of radius $R$”. For example, $S_3(R)$ is the sphere of radius $R$; $S_2(R)$ is the (interior of the) circle of radius $R$; $S_1(R)$ is a line segment of length $2R$ (why?).

1. Make a table of values of the “volume” $V_n(R)$ and “surface area” $A_n(R)$ of $S_n(R)$ for $n = 2, 3, 1$ and $0$. (This is an example of “generalizing downward”, and will require some creative thinking.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$V_n(R)$ = “volume” of $S_n(R)$</th>
<th>$A_n(R)$ = “surface area” of $S_n(R)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
<td>1</td>
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<tr>
<td>5</td>
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</tbody>
</table>

2. Why is $S_n(R)$ a constant multiple of $R^n$? Why is $A_n(R)$ a constant multiple of $R^{n-1}$?

3. Why is $A_n(R) = \frac{d}{dR} S_n(R)$?

In fact,

\[
S_n(R) = \frac{\pi^{n/2}}{(n/2)!} R^n.
\]

4. “But wait!” you exclaim — “we don’t know the meaning of $(n/2)!$ when $n$ is odd!” So using the table, define $(1/2)!$. Then define $n! = n(n - 1)!$ even when $n$ is a half-integer.

Date: Tuesday, November 19, 2002.
Check your answer by verifying that the resulting formula for the volume of the 3-sphere still works.

5. Prove equation (1) by induction as follows. Prove it for \( n = 1 \) and 2. Then prove it for \( n \) assuming it holds for \( n - 2 \), by showing that

\[
\text{vol } S_n(R) = \iiint_{\vec{x} \in n\text{-sphere of radius } R} 1 = \iiint_{\vec{y} \in \text{circle of radius } R} \left( \iint_{\vec{z} \in (n-2)\text{-sphere of radius } \sqrt{R^2 - |\vec{y}|^2}} 1 \right)
\]

and computing the nested integrals on the right. (At some point, polar coordinates may help.)

6. (This is easier than many of the earlier ones.) There is a less ad hoc definition of \( n! \) when \( n \) isn’t an integer. If \( s \) is a non-negative real number, define \( g(s) = \int_0^\infty x^s e^{-x} dx \). Prove that (i) \( g(0) = 1 \), (ii) \( g(s) = sg(s - 1) \) if \( s \geq 1 \), assuming that \( g(s - 1) \) exists. Be careful with convergence! Hence \( g(s) = s! \) when \( s \) is an integer.

This function, shifted by one, is called the “gamma function”. There is a neat argument that \( g(1/2) \) is the value you found in problem 5; thus \( g(s) = s! \) even when \( s \) is a half-integer.

Putnam Problems.

1987A3. For all real \( x \), the real-valued function \( y = f(x) \) satisfies

\[
y'' - 2y' + y = 2e^x.
\]

(a) If \( f(x) > 0 \) for all real \( x \), must \( f'(x) > 0 \) for all real \( x \)? Explain.
(b) If \( f'(x) > 0 \) for all real \( x \), must \( f(x) > 0 \) for all real \( x \)? Explain.

1987B1. Evaluate

\[
\int_2^4 \frac{\sqrt{\ln(9 - x)}}{\sqrt{\ln(9 - x)} + \sqrt{\ln(9 + x)}} dx.
\]

1995A2. For what pairs \((a, b)\) of positive real numbers does the improper integral

\[
\int_b^\infty \left( \sqrt{\sqrt{x + a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x + b}} \right) dx
\]

converge?

1991B2. Suppose \( f \) and \( g \) are nonconstant, differentiable, real-valued functions on \( \mathbb{R} \). Furthermore, suppose that for each pair of real numbers \( x \) and \( y \),

\[
\begin{align*}
f(x + y) &= f(x)f(y) - g(x)g(y), \\
g(x + y) &= f(x)g(y) + g(x)f(y).
\end{align*}
\]

If \( f'(0) = 0 \), prove that \((f(x))^2 + (g(x))^2 = 1\) for all \( x \).
Let $f$ be a twice-differentiable real-valued function satisfying
\[ f(x) + f''(x) = -xg(x)f'(x), \]
where $g(x) \geq 0$ for all real $x$. Prove that $|f(x)|$ is bounded.

Let $f$ be a real function on the real line with continuous third derivative. Prove that there exists a point $a$ such that
\[ f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0. \]

Find the maximum value of
\[ \int_0^y \sqrt{x^4 + (y - y^2)^2} \, dx \]
for $0 \leq y \leq 1$.

Show that
\[ \int_{-100}^{-10} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \, dx + \int_{10}^{101} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \, dx + \int_{101}^{1000} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \, dx \]
is a rational number.

Find the set of all real numbers $k$ with the following property: For any positive, differentiable function $f$ that satisfies $f'(x) > f(x)$ for all $x$, there is some number $N$ such that $f(x) > e^{kx}$ for all $x > N$.

Evaluate
\[ \int_0^\infty \left( x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left( 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) \, dx. \]

This handout can (soon) be found at
\[ http://math.stanford.edu/~vakil/stanfordputnam/ \]

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