1. Where we are: Segre classes of vector bundles, and Segre classes of cones

We first defined Segre class of vector bundles over an arbitrary scheme $X$. If $E$ is a vector bundle, we get an operator on class on $X$. We define it by projectivizing $E$, so we have a flat and proper morphism $\mathbb{P}E \rightarrow X$, pulling back $\alpha$ to $\mathbb{P}E$, capping with $O(1)$ a certain number of times, and pushing forward.

Hence we get $s_i(E) \cap : A_kX \rightarrow A_{k-i}X$, and for example we checked the non-immediate fact that $s_0(E)$ is the identity. (Recall $s_0$ involved pulling back, capping with precisely $\text{rank } E - 1$ copies of $O(1)$, and then pushing forward.) Note that $s_k(E) = s_k(E \oplus 1)$, as the Whitney product formula gives $s(E \oplus 1) = s(E)s(1) = s(E)$.

We want to generalize this to cones. Here again is the definition of a cone on a scheme $X$. Let $S = \oplus_{i \geq 0} S^i$ be a sheaf of graded $O_X$-algebras. Assume $O_X \rightarrow S^0$ is surjective, $S^1$ is coherent, and $S^i$ is generated (as an algebra) by $S^1$. Then you can define $\text{Proj}(S^i)$, which has a line bundle $O(1)$. $\text{Proj}(S^i) \rightarrow X$ is a projective (hence proper) morphism, but it isn’t necessarily flat! (Draw a picture, where the cone has components of different dimension.) Flat morphisms have equidimensional fibers, and cones needn’t have this.

A couple of important points, brought out by Joe and Soren. I’ve been imprecise with terminology. Although one often sees phrases such as “the cone is $C = \text{Spec}(S^i)$”, we lose a little information this way; the cone should be defined to be the graded sheaf $S^i$. The sheaf can be recovered from $C_XY$ along with the action of the multiplicative group $O^*_{X,Y}$; the nth graded piece is the part of the algebra where the multiplicative group acts with weight $n$.

Example 1: say let $E$ be a vector bundle, and $S^i = \text{Sym}^i(E^\vee)$. Then $\text{Proj} S^i = \mathbb{P}E$. Example 2: Say $T^i = \text{Sym}^i(E^\vee \oplus 1) = S^i \oplus S^{i-1}z$, so (better) $T^i = S^i[z]$. Then $\text{Proj} T^i = \mathbb{P}E$. Example 3:
Proj$(S[z]) = C \coprod \text{Proj}(S) = \text{Spec } S \coprod \text{Proj}(S)$. The argument is just the same. The right term is a Cartier divisor in class $O_{\text{Proj}(S[z])}(1)$. Example 4: The blow-up can be described in this way, and it will be good to know this. Suppose $X$ is a subscheme of $Y$, cut out by ideal sheaf $I$. (In our situation where all schemes are finite type, $I$ is a coherent sheaf.) Then let $S = \oplus I^i$, where $I$ is the $i$th power of the ideal $I$. ($I^0$ is defined to be $O_X$.) Then $\text{Bl}_X Y \cong \text{Proj}(S)$. A short calculation shows that the exceptional divisor class is $O(-1)$.

The exceptional divisor turns out to be $\text{Proj}(\oplus I^i/I^{i+1})$. (Note that this is indeed a graded sheaf of algebras.) As $\oplus I^i \to \oplus I^{i+1}$ is a surjective map of rings, this indeed describes a closed subscheme of the blow-up. (Remember this formula — it will come up again soon!)

So the same construction of Segre classes of vector bundles doesn’t work: there is no flat pullback to $\text{Proj}(S)$. So what do we do?

Idea (slightly wrong): We can’t pull classes back to $\text{Proj}(S)$. But there is a natural class up there already: the fundamental class. So we define

$$s(C) = q_*(\sum_{i \geq 0} c_1(O(1))^i \cap [\text{Proj}(C)])$$

where $q$ is the morphism $\text{Proj}(C) \to X$. Instead, as Segre class of vector bundles are stable with respect to adding trivial bundles, we define

$$s(C) := q_*(\sum_{i \geq 0} c_1(O(1))^i \cap [\text{Proj}(C \oplus 1)])$$

where $q$ is the morphism $\text{Proj}(C \oplus 1) \to X$. Why is adding in this trivial factor the right thing to do? Partial reason: if $C$ is the 0 cone, i.e. $S^i = 0$ for $i > 0$, then $\text{Proj}(C)$ is empty, but $\text{Proj}(C \oplus 1)$ is not; we get different answers. But if you add more 1’s, you will then get the same answer: $s(C \oplus 1 \oplus \cdots \oplus 1) = s(C)$.

(Exercise: show that $s(C \oplus 1) = s(C)$.)

Note: $s$ has pieces in various dimensions.

Last time I proved:

**Proposition.** (a) If $E$ is a vector bundle on $X$, then $s(E) = c(E)^{-1} \cap [X]$, where $c(E)$ is the total Chern class of $X$, $r = \text{rank}(E)$. $c(E) = 1 + c_1(E) + \cdots + c_r(E)$. (I would write $s(E) = s(E) \cap [X]$, but the two uses of $s(E)$ are confusing!) This is basically our definition of Segre/Chern classes.

(b) Let $C_1, \ldots, C_1$ be the irreducible components of $C$, $m_i$ the geometric multiplicities of $C_i$ in $C$. Then $s(C) = \sum_{i=1}^r m_i s(C_i)$. (Note that the $C_i$ are cones as well, so $s(C_1)$ makes sense.) In other words, we can compute the Segre class piece by piece.
2. THE NORMAL CONE, AND THE SEGRE CLASS OF A SUBVARIETY

Let \( X \) be a closed subscheme of a scheme \( Y \) (not necessarily lci = local complete intersection), cut out by ideal sheaf \( \mathcal{I} \).

\( \mathcal{I}/\mathcal{I}^2 \) is the conormal sheaf to \( X \); it is a sheaf on \( X \). (Why is it a sheaf on \( X \)? Locally, say \( Y = \text{Spec} \, R \) and \( X = \text{Spec} \, R/\mathfrak{I} \). Then this is the \( R \)-module \( \mathcal{I}/\mathcal{I}^2 \). The fact that I said that it is an \( R \)-module makes it a priori a sheaf on \( Y \). But note that it is also an \( R/\mathfrak{I} \) module; the action of \( \mathfrak{I} \) on \( \mathcal{I}/\mathcal{I}^2 \) is the zero action.) If \( X \) is a local complete intersection (regular imbedding), then this turns out to be a vector bundle.

Consider \( \sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1} \). (Recall that \( \text{Proj} \) of this sheaf gives us the exceptional divisor of the blow-up.) Define the normal cone \( C = C_{X Y} \) by

\[
C = \text{Spec} \sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}.
\]

Define the Segre class of \( X \) in \( Y \) as the Segre class of the normal cone:

\[
s(X, Y) = s(C_{X Y}) \in \mathbb{A}_X.
\]

If \( X \) is regularly imbedded (=lci) in \( Y \), then the definition of \( s(X, Y) \) is

\[
s(X, Y) = s(N) \cap [X] = c(N)^{-1} \cap [X].
\]

The following geometric picture will come up in the central construction in intersection (the deformation to the normal cone). \( X \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{A}^1 \). Then blow up \( X \times 0 \) in \( Y \times \mathbb{A}^1 \). The ideal sheaf of \( X \times 0 \) is \( \mathcal{I}[t] \), where \( t \) is the coordinate on \( \mathbb{A}^1 \). Thus the normal cone to \( X \times 0 \) in \( Y \times \mathbb{A}^1 \) is \( C_{X Y}[t] \). Hence the exceptional divisor is \( \text{Proj}(C_{X Y}[t]) \) (draw a picture). Inside it is the Cartier divisor \( t = 0 \), which is \( \text{Proj}(C_{X Y}) \).

3. SEGRE CLASSES BEHAVE WELL WITH RESPECT TO PROPER AND FLAT MORPHISMS

This is the key result of the chapter.

**Proposition.** Let \( f : Y' \to Y \) be a morphism of pure-dimensional schemes, \( X \subset Y \) a closed subscheme, \( X' = f^{-1}(X) \) the inverse image scheme, \( g : X' \to X \) the induced morphism.

(a) If \( f \) proper, \( Y \) irreducible, and \( f \) maps each irreducible component of \( Y' \) onto \( Y \) then

\[
g_*(s(X', Y')) = \deg(Y'/Y)s(X, Y).
\]

(b) If \( f \) flat, then

\[
g^*(s(X', Y')) = s(X, Y).
\]

Let me repeat why I find this a remarkable result. \( X' \) is a priori some nasty scheme; even if it is nice, its codimension in \( Y' \) isn’t necessarily the same as the codimension of \( X \) in \( Y \). The argument is quite short, and shows that what we’ve proved already is quite sophisticated.
As a special case, this result shows that Segre classes have a fundamental birational invariance: if \( f : Y' \to Y \) is a birational proper morphism, and \( X' = f^{-1}X \), then \( s(X', Y') \) pushes forward to \( s(X, Y) \).

**Proof.** Let me assume that \( Y' \) is irreducible. (It’s true in general, and I may deal with the general case later.)

Let me first write the diagram on the board, and then explain it.

\[
\begin{array}{c}
\mathcal{O}_{\text{Proj}(C' \oplus 1)}(1) \\
\mathcal{O}_{\text{Proj}(C \oplus 1)}(1) \\
\text{Proj}(C' \oplus 1)^{\text{Cartier div}} \xrightarrow{\text{Bl}_{X' \times 0}} (Y' \times \mathbb{A}^1) \\
\text{Proj}(C \oplus 1)^{\text{Cartier div}} \xrightarrow{\text{Bl}_{X \times 0}} (Y \times \mathbb{A}^1) \\
X' \xrightarrow{g} X \\
X' \xrightarrow{q} \text{Proj}(C \oplus 1) \\
\end{array}
\]

We blow up \( Y \times \mathbb{A}^1 \) along \( X \times 0 \), and similarly for \( Y' \) and \( X' \). The exceptional divisor of \( \text{Bl}_{X \times 0}(Y \times \mathbb{A}^1) \) is \( \text{Proj}(C \oplus 1) \), and similarly for \( Y' \) and \( X' \). The universal property of blowing up \( Y \times \mathbb{A}^1 \) shows that there exists a unique morphism \( G \) from the top exceptional divisor to the bottom. Moreover, by construction, the exceptional divisor upstairs is the pullback of the exceptional divisor downstairs (that’s the statement about the two \( \mathcal{O}(1) \)'s in the diagram). Let \( q \) be the morphism from the exceptional divisor \( \text{Proj}(C \oplus 1) \) to \( X \), and similarly for \( q' \). That square commutes: \( q \circ G = g \circ q' \) (basically because that morphism \( G \) was defined by the universal property of blowing up).

Now \( f_*(Y' \times \mathbb{A}^1) = d(Y \times \mathbb{A}^1) \) (where I am sloppily using the name \( f \) for the morphism \( Y' \times \mathbb{A}^1 \to Y \times \mathbb{A}^1 \)). This is computed on a dense open set, so blow-up doesn’t change this fact:

\[
F_*[\text{Bl}_{X' \times 0} Y' \times \mathbb{A}^1] = d[\text{Bl}_{X \times 0} Y \times \mathbb{A}^1].
\]

Now we’ve shown that proper pushforward commutes with intersecting with a (pseudo-)Cartier divisor. Hence

\[
G_*[\text{Proj}(C' \oplus 1)] = d[\text{Proj}(C \oplus 1)].
\]
Now I’m going to prove (a), and I’m going to ask you to prove (b) with me, so pay attention!

\[ g_*s(X', Y') = g_*q'_* \left( \sum_i c_1(G^*(\mathcal{O}(1)) \cap \mathbb{P}(C' \oplus 1)) \right) \text{ (by def’n)} \]

\[ = q_*G_* \left( \sum_i c_1(G^*(\mathcal{O}(1)) \cap \mathbb{P}(C' \oplus 1)) \right) \text{ (prop. push. commute)} \]

\[ = q_* \left( \sum_i c_1((\mathcal{O}(1)) \cap d[\mathbb{P}(C \oplus 1)]) \right) \text{ (proj. form.)} \]

\[ \text{(i.e. } c_1 \text{ commutes with prop. pushforward)} \]

\[ = ds(X, Y) \text{ (by def’n)} \]

Now (b) is similar:

\[ g^*s(X, Y) = g^*q_* \left( \sum_i c_1((\mathcal{O}(1)) \cap \mathbb{P}(C \oplus 1)) \right) \text{ (by def’n)} \]

\[ = q'_*G^* \left( \sum_i c_1((\mathcal{O}(1)) \cap \mathbb{P}(C \oplus 1)) \right) \text{ (push/pull commute)} \]

\[ = q'_* \left( \sum_i c_1((G^*(\mathcal{O}(1)) \cap G^*[\mathbb{P}(C \oplus 1)]) \right) \]

\[ = s(X, Y) \text{ (by def’n)} \]

We immediately have:

**Corollary.** With the same assumptions as the proposition, if \( X' \) is *regular imbedded* (=lci) in \( Y' \), with normal bundle \( N' \), then

\[ g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)s(X, Y). \]

If \( X \subset Y \) is also regularly imbedded, with normal bundle \( N \), then

\[ g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)(c(N)^{-1} \cap [X]). \]

To see why the first part might matter: Suppose \( X \hookrightarrow Y \) is a very nasty closed immersion. Then blow up \( Y \) along \( X \), to get \( Y' \) with exceptional divisor \( X' \). Then \( X' \) is regularly imbedded (lci) in \( Y' \) — it is a Cartier divisor! This is the content of the next corollary.
Corollary. Let $X$ be a open closed subscheme of a variety $Y$. Let $\tilde{Y}$ be the blow-up of $Y$ along $X$, $\tilde{X} = \mathbb{P}C$ the exceptional divisor, $\eta : \tilde{X} \rightarrow X$ the projection. Then

$$s(X, Y) = \sum_{k \geq 1} (-1)^{k-1} \eta^*(\tilde{X}^k)$$

$$= \sum_{i \geq 0} \eta^*(c_1(O(1))^i \cap [\mathbb{P}C])$$

In that first equation, the term $\tilde{X}^k$ should be interpreted as the $k$th self intersection of the Cartier divisor $\tilde{X}$, also known as the exceptional divisor.

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