Recap of last time. Last time we began discussing blow-ups:

Given \( p \in S \), there is a surface \( S' = Bl_p S \) and a morphism \( \pi : S' \to S \), unique up to isomorphism, such that (i) the restriction of \( \pi \) to \( \pi^{-1}(S - \{p\}) \) is an isomorphism onto \( S - \{p\} \), and (ii) \( \pi^{-1}(p) \) is isomorphic to \( \mathbb{P}^1 \). \( \pi^{-1}(p) \) is called the exceptional divisor \( p \), and is called the exceptional divisor.

A key example, and indeed the analytic-, formal-, or etale-local situation, was given by blowing up \( S = \mathbb{A}^2 \) at the origin, which I’ll describe again soon when it comes up in a proof.

For the definition, complex analytically, you can take the same construction. Then you need to think a little bit about uniqueness. There is a more intrinsic definition that works algebraically, let \( \mathcal{I} \) be the ideal sheaf of the point. Then \( S' = \text{Proj} \bigoplus_{d \geq 0} \mathcal{I}^d \).

1. How basic aspects of surfaces change under blow-up

Definition. If \( C \) is a curve on \( S \), define the strict transform \( C^{\text{strict}} \) of \( C \) to the the closure of the pullback on \( S - p \), i.e. \( \pi|_{S' - E}^{-1}(C \cap S - p) \). The proper transform \( C^{\text{proper}} \) is given by the pullback of the defining equation, so for example \( \pi^* \mathcal{O}_S(C) = \mathcal{O}_{S'}(C') \).

Lemma. If \( C \) has multiplicity \( m \) at \( p \), then \( C^{\text{proper}} = C^{\text{strict}} + mE \), i.e. \( \pi^* C \cong C^{\text{strict}} + mE \).

Proof. The multiplicity of \( C \) being \( m \) means that in local coordinates, the defining equation has terms of degree \( m \), but not lower. (Better: the defining equation lies in \( m^m \) but not \( m^{m+1} \).) Analytically, this means that the leading term in \( x \) and \( y \) has degree \( m \).
We do this by local calculation which will be useful in general. (Draw picture.) $U_0 = \{ ((x_0, y_0), [1; v]) : y_0 = x_0 v \} = \text{Spec } k[x_0, y_0, v]/y_0 = x_0 v = \text{Spec } k[x_0, v]$. The exceptional divisor $E$ is given by $y_0 = 0$ (after morphism).

$$U_1 = \{ ((x_1, y_1), [u; 1]) : x_1 = y_1 u \} = \text{Spec } k[x_1, y_1, u]/x_1 = y_1 u = \text{Spec } k[y_1, u].$$ The exceptional divisor $E$ is given by $y_1 = 0$.

Map down to $(x, y) = \text{Spec } k[x, y]. (x_0, y_0, v) \mapsto (x_0, y_0), (x_1, y_1, v) \mapsto (x_1, y_1).

Given a function $f(x, y) = 0$. Pull it back to $U_0$: $f(x, y) = f_m(x, y) + \text{higher} = f_m(x_0, x_0 v) + \text{higher} + \cdots$. $\square$

**Exercise.** To see if you understood that, do the same calculation on patch 2.

**Theorem.** Suppose $\pi : S' \to S$ is a blow up of $S$ at $p$, with exceptional curve $E \subset S'$. Let $D$ and $D'$ be divisors on $S$. Then $\pi^* D \cdot \pi^* D' = D \cdot D' , E \cdot \pi^* D = 0, E^2 = -1$.

**Remark.** A curve on a smooth surface that is isomorphic to $\mathbb{P}^1$ and has self-intersection $-1$ is called a $(-1)$-curve.

**Proof.** The first we did yesterday. The second: by Serre’s moving lemma, we can move $D$ away from $p$, then pull back. For the third: choose a curve $C$ passing through $p$ with multiplicity $1$. (How to do this: hyperplane section of $S$.) Then $C_{\text{strict}} \cdot E = 1$. Also $C_{\text{proper}} \cdot E = 0$. as $C_{\text{strict}} + E = C_{\text{proper}}$, we’re done.

**Theorem.** (a) There is an isomorphism $\text{Pic } S \oplus \mathbb{Z} \cong \text{Pic } S'$ defined by $(D, n) \mapsto nE + \pi^* D$. (b) The same with $\text{Pic}$ replaced by $NS$.

**Proof.** The arguments are the same for both parts, so I’ll do (a). It is surjective: the divisors upstairs are either $E$ or strict transforms (which are proper transforms plus $E$’s). It is injective: if $\pi^* D + nE = 0$, then intersect with $E$ to see that $n = 0$; then apply $\pi_*$ to see that $D = 0$.

**Theorem.** $K_{S'} = \pi^* K_S + E$.

**Proof.** Clearly $K_{S'} = \pi^* K_S + mE$ for some $m$. By the adjunction formula for $E$, $K_E = K_{S'}(E)|_E$. Taking degrees:

$$-2 = (\pi^* K_S + mE + E) \cdot E = -m - 1.$$

$\square$

**Exercise/Remark.** If you want practice with the canonical bundle in local coordinates, take a meromorphic section of $K_S$ that has neither zero nor pole at $p$ (possible by Serre’s moving lemma), write it as $f(x, y)dx \wedge dy$, and pull it back to the open set $U_1$ to see that you get $f(x_0, x_0 v)dx_0 \wedge dv = f(x_0, x_0 v)x_0 dx_0 \wedge dv$. 
2. RATIONAL MAPS OF SURFACES, LINEAR SYSTEMS, AND ELIMINATION OF INDETERMINACY

A rational map \( S \to X \), where \( X \) is a variety, means a morphism from an dense open set of \( S \). Recall that a rational map from a curve \( C \) to a projective variety can always be extended to a morphism. Similarly, a rational map from a surface \( S \) to a projective variety can be extended over most points; the set of indeterminacy is a finite set of points. More precisely, given a map \( \pi : S \to \mathbb{P}^n \). This is given by \( n + 1 \) sections of some line bundle. It makes sense except where the sections are all zero. This will be in codimension 2.

Let \( F \) be this finite set. We’ll denote \( \overline{\pi(S - F)} \) the image of \( S \), and denote it \( \pi(S) \). (I’m not sure we need to take the closure.) If \( C \) is a curve on \( S \), then we’ll denote \( \overline{\pi(C - F)} \) the image of \( C \), and denote it \( \pi(C) \). Here we definitely need to take the closure.

Now suppose you have a divisor \( D \) on \( S \). Given a subspace \( V \) of dimension \( n \) of \( H^0(S, \mathcal{O}(D)) \), we might hope to get a map to projective space \( \mathbb{P}^V \). (This is called a linear system of dimension \( n \); I should have introduced this notation earlier.) If it is base point free, we do.

If it has base points, the locus could have components of dimension 1. Such a component is called a fixed component of the linear system \( V \). The fixed part of \( V \) is the biggest divisor contained in every element of \( V \). So if this fixed part is \( F \), then \( D - F \) has no fixed components.

(I’m not happy with how I explained the previous paragraphs in class. I hope this is clearer.)

**Lemma.** If the linear systems has no fixed part, then it has only a finite number of fixed points.

**Proof.** Take two general sections, and look at their two zero-sets. Where do they intersect? At a bunch of points. Hence we get at most \( D^2 ? \)

We’ve basically shown that there is a bijection between:

(i) \{ rational maps \( \pi : S \to \mathbb{P}^n \) such that \( \pi(S) \) is contained in no hyperplane \}

(ii) \{ linear systems on \( S \) without fixed part and of dimension \( n \) \}

(Explain the correspondence.)

**Theorem (Elimination of indeterminacy).** Let \( \pi : S \to X \) be a rational map from a surface to a projective variety. Then there exists a surface \( S' \), a morphism \( \eta : S' \to S \) which is the composite of a finite number of blow-ups, and a morphism \( f : S' \to X \) such
that the diagram

\[
\begin{array}{c}
S' \\
\downarrow \eta \\
S \\
\downarrow \pi \\
X
\end{array}
\]

is commutative.

**Proof.** Idea: blow up fixed points, show that \( D^2 \) decreases.

We immediately reduce to the case where \( X \) is \( \mathbb{P}^m \), and \( \pi(S) \) isn’t contained in any hyperplane of \( \mathbb{P}^m \). Then \( \phi \) corresponds to a linear system \( V \subset |D| \) of dimension \( n \) on \( \tilde{S} \), with no fixed component. If \( V \) has no base point, then we’re done.

Otherwise, we blow up a base point \( x \), and consider \( S_1 \to S \) at \( x \) (and hence a rational map \( S_1 \to S \)). The exceptional curve is now in the fixed part of the linear system, with some multiplicity \( k \geq 1 \). So we subtract \( kE \) to get rid of the fixed part, i.e. get a new linear system \( V_1 \subset |\pi^*D - kE| \), to get the same rational map \( \phi_1 : S_1 \to S \), given by \( D_1 = D - kE \). If this is a morphism, we win, otherwise we keep going.

At some point, this process must stop (and hence we win in the long run). We prove this is the case when \( D^2 = i \), by induction on \( i \). Base case, \( i = 0 \): the number of fixed points is bounded by \( D^2 = 0 \), so there aren’t any. Inductive step: Now \( i > 0 \). Then we blow-up once, and we get a new surface with divisor class. On this surface, \( D^2_1 = (D - kE)(D - kE) = D^2 - k^2 < D^2 \). So by the inductive hypothesis, the process will terminate on this new surface, completing the induction.

### 3. The Universal Property of Blowing Up

**Theorem (Universal property of blowing up).** Let \( f : X \to S \) be a birational morphism of surfaces, and suppose that the rational map \( f^{-1} \) is undefined at a point \( p \) of \( S \). Then \( f \) factorizes as

\[
f : X \to \tilde{S} := \text{Bl}_p S \to S
\]

where \( g \) is a birational morphism and \( \pi \) is the blow-up at \( p \).

Proof: next day.

#### 3.1. Applications of the universal property of blowing up. Two theorems.

**Theorem (all birational morphisms factor into blow-ups).** Let \( f : S \to S_0 \) be a birational morphism of surfaces. Then there is a sequence of blow-ups \( \pi_k : S_k \to S_{k-1} \) \((k = 1, \ldots, n)\) and an isomorphism \( u : S \to S_n \) such that \( f = \pi_1 \circ \cdots \circ \pi_n \circ u \).

**Proof.** If \( f \) is an isomorphism, we’re done. Otherwise, there is a point \( p \) of \( S_0 \) such that \( f^{-1} \) is undefined at \( p \), and we can factor through \( S \to S_1 = \text{Bl}_p S_0 \). We can repeat this.

If \( n(f_k) \) is the number of contracted curves of \( n(f_k) < n(f_{k+1}) \): if \( E \) is the exceptional divisor of \( \pi_k : S_k \to S_{k-1} \), then the preimage of \( E \) in \( S \) contains a curve which is contracted
by $f_{k-1}$ but not $f_k$. As the number of contracted curves can’t be negative, the process must terminate.

**Theorem (all birational maps can be factored into blow-ups).** Let $\phi : S \to S'$ be a birational map of surfaces. Then there is a surface $S''$ and a commutative diagram

$$
\begin{array}{ccc}
S'' & \xrightarrow{g} & S' \\
\downarrow f & & \downarrow \phi \\
S & \xrightarrow{\phi} & S'
\end{array}
$$

where the morphisms $f$ and $g$ are composites of blow-ups.

**Proof.** By the theorem of elimination of indeterminacy, we can find such a diagram such that $f$ is a composition of blow-ups. By the Theorem above, $g$ must then be a composition of blow-ups too.

We’ve now proved some powerful stuff, so let’s take a step back and see what we now know, and how it relates to classification.

Two surfaces are birational iff they can be be related by sequences of blow-ups. We’ll be interested in birational classification, but biregular classification is very close.

If $f : S \to S'$ is birational which is the composition of $n$ blow-ups, then $NS(S) \cong NS(S') \oplus \mathbb{Z}^n$, so $n$ is independent of the choice of blow-ups. **Exercise:** Use this to show that every birational morphism from $S$ to itself is an isomorphism.

**Fact.** In a blow-up, $H^i$ of the structure sheaf is preserved, i.e. if $\pi : S' \to S$ is a blow-up, then $\pi^* : H^i(\mathcal{O}_S) \to H^i(\mathcal{O}_{S'})$ is an isomorphism.

The algebraic way of proving this fact comes from the Leray spectral sequence, and the fact that $\pi_* \mathcal{O}_{S'} = \mathcal{O}_S$ and $R^i \pi_* \mathcal{O}_{S'} = 0$ for $i > 0$. This in turn requires some infinitesimal analysis, in the form of “formal function theorems”. I suspect that there should be a relatively straightforward analytic proof.

In particular, by these numbers are birational invariants.

So look at what this means for the Hodge diamond. When you blow up, you add 1 to the central entry (the rank of the Neron-Severi group). Everything else is constant.

**Next day:** More consequences of these powerful theorems. Proof of the universal property of blowing up. Castelnuovo’s criterion for blowing down curves.