Recap of last time. Extended example of $\mathbb{P}^n$. $O_{\mathbb{P}^n}(d) \leftrightarrow$ degree $d$ homogeneous things.

Maps to projective space corresponded to vector spaces of sections of an invertible sheaf $\mathcal{L}$ that are basepoint free (no common zero). Hyperplane sections correspond to $H^0(X, \mathcal{L})$.

For example, if the sections of $\mathcal{L}$ have no common zero, then we can map to some projective space by the vector space of all sections. Then we say that the invertible sheaf is $\mathcal{L}$ is basepoint free. (I didn’t give this definition last time.)

Definition. The corresponding map to projective space is called a linear system. (I’m not sure if I’ll use this terminology, but I want to play it safe.)

$$|\mathcal{L}| : X \to \mathbb{P}^n = \mathbb{P}H^0(X, \mathcal{L})^*.$$ 

Definition. An invertible sheaf $\mathcal{L}$ is very ample if the global sections of $\mathcal{L}$ gives a closed immersion into projective space.

Fact. equivalent to: “separates points and tangent vectors”.

Definition. An invertible sheaf is ample if some power of it is very ample.

Note: A very ample sheaf on a curve has positive degree. Hence an ample sheaf on a curve has positive degree. We’ll see later today that this is an “if and only if”.

Date: Friday, October 11.
**Fact (Serre vanishing).** Suppose $\mathcal{M}$ is any coherent sheaf e.g. an invertible sheaf, or more generally a locally free sheaf (essentially, a vector bundle), and $\mathcal{L}$ is ample. Then for $n >> 0$, $H^i(X, \mathcal{M} \otimes \mathcal{L}^n) = 0$ for $i > 0$.

1. **SERRE DUALITY AND RIEMANN-ROCH; BACK TO CURVES**

**Fact: Serre duality.** If $X$ is proper nonsingular and dimension $n$, then for $0 \leq i \leq n$,

$$H^i(X, \mathcal{L}) \otimes H^{n-i}(X, \mathcal{K} \otimes \mathcal{L}^\vee) \to H^n(X, \mathcal{K}) \sim \mathbb{C}$$

is a perfect pairing.

(True for vector bundles. More general formulation for arbitrary coherent sheaves.) Thus $h^i(X, \mathcal{L}) = h^{n-i}(X, \mathcal{K} \otimes \mathcal{L}^*)$ and $\chi(C, \mathcal{L}) = (-1)^n\chi(C, \mathcal{K} \otimes \mathcal{L}^\vee)$.

In particular, we have **Serre duality for curves**. For any invertible sheaf $\mathcal{L}$, the map

$$H^0(C, \mathcal{L}) \otimes H^1(C, \mathcal{K} \otimes \mathcal{L}^\vee) \to H^1(C, \mathcal{K}) \sim \mathbb{C}$$

is a perfect pairing.

Hence two possible definitions of genus are the same: $h^0(C, \mathcal{K})$ and $h^1(C, \mathcal{O}_C)$.

**Fact: Riemann-Roch Theorem.**

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg(\mathcal{L}) - g + 1.$$  

(I.e. $\chi(C, \mathcal{L}) = \deg -g + 1$ — remember that the cohomology of a coherent sheaf vanishes above the dimension of the variety.)

Generalizations: Hirzebruch-Riemann-Roch, to Grothendieck-Riemann-Roch. Hirzebruch-Riemann-Roch, which we'll be using, is a consequence of the Atiyah-Singer index theorem, which Rafe Mazzeo spoke about in the colloquium yesterday.

Proof of Riemann-Roch: (i) algebraic, (ii) complex-analytic, (iii) Atiyah-Singer.

Using Serre duality:

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{K} \otimes \mathcal{L}^\vee) = \deg(\mathcal{L}) - g + 1.$$  

**Corollary.** $\deg K = 2g - 2$. (Do it.)

So we have another definition of $g$.

2. **APPLICATIONS OF RIEMANN-ROCH**

**Theorem.** If $\deg \mathcal{L} \geq 2g - 1$, then $h^1(C, \mathcal{L}) = 0$. Hence $h^0(C, \mathcal{L}) = \deg \mathcal{L} - g + 1$.

Proof: Serre duality, and the fact that invertible sheaves of negative degree have no sections. Then Riemann-Roch.
**Theorem (numerical criterion for basepoint freeness).** If $\deg \mathcal{L} \geq 2g$, then $\mathcal{L}$ is basepoint free. (Remind them of definition.)

**Proof:** $H^0(C, \mathcal{L}(-p))$ is the vector space of sections with a zero at $p$. Want to show that $H^0(C, \mathcal{L}) - H^0(C, \mathcal{L}(-p)) > 0$. Do it using previous result: difference is 1!

**Example.** If $C$ is genus 1 and $\mathcal{L}$ is degree 2, then we get a map to $\mathbb{P}^1$. This shows that every genus 1 curve is a double cover of $\mathbb{P}^1$. (Explain “hyperplane section” in this case.)

2.1. **Classification of genus 2 curves. Theorem.** Any genus 2 curve has a unique double cover of $\mathbb{P}^1$ branched over 6 points. (The 6 points are unique up to automorphisms of $\mathbb{P}^1$.) Any double cover of $\mathbb{P}^1$ branched over 6 points comes from a genus 2 curve. Hence genus 2 curves are classified by the space of 6 distinct points of $\mathbb{P}^1$, modulo automorphisms of $\mathbb{P}^1$. In particular, there is a dimension $6 - \dim \text{Aut } \mathbb{P}^1 = 3$ moduli space.

**Proof.** First, every genus 2 curve $C$ has a degree 2 map to $\mathbb{P}^1$ via $\mathcal{K}$ (basepoint freeness). The numerical criterion for basepoint freeness doesn’t apply, unfortunately. Let’s check that $\mathcal{K}$ has 2 sections: $h^0(\mathcal{K}) - h^0(\mathcal{O}) = \deg(\mathcal{K}) - 2 + 1 = 1$. Next, let’s check that $\mathcal{K}$ is basepoint free. We want $\mathcal{K}(-p)$ has only 1 section for all $p$. If $\mathcal{K}(-p)$ had two sections, then we’d have a degree 1 map to $\mathbb{P}^1$ from $C$ (explain), contradiction.

Next we’ll see that the only degree 2 maps to $\mathbb{P}^1$ arise from the canonical bundle. Suppose $\mathcal{L}$ is degree 2, and has 2 sections. Then by RR, $\mathcal{K} \otimes \mathcal{L}^*$ has 1 section, and is degree 0. But the only degree 0 sheaf with a section is the trivial sheaf.

Finally, any double cover of $\mathbb{P}^1$ branched over 6 points is genus 2. This follows from the Riemann-Hurwitz formula.

Using naive geometry: $2\chi(\mathbb{P}^1) - 6 = \chi(C)$, i.e. $-2 = 2 - 2g$, so $g = 2$.

**Fact.** Riemann-Hurwitz formula: $\mathcal{K}_C = \pi^* \mathcal{K}_{\mathbb{P}^1} + \text{ramification divisor}$. Taking degrees: $2g - 2 = 2(-2) + 6$. $\square$

2.2. **A numerical criterion for very ampleness. Theorem.** If $\deg \mathcal{L} \geq 2g + 1$, then $\mathcal{L}$ is very ample. (Remind them of definition.)

**Proof.** I’ll show you that it separates points. Suppose $p \neq q$. $H^0(C, \mathcal{L}) - H^0(C, \mathcal{L}(-p - q)) = 2$. Hence there is a section vanishing at $p$ and not at $q$, and vice versa.

Same idea works for separating tangent vectors. $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 2$. Hence there is a section vanishing at $p$ but not to order 2.

**Example.** $C$ genus 1, $\mathcal{L}$ has degree 3. Then $C \hookrightarrow \mathbb{P}^2$, degree 3. Hence every elliptic curve can be described by a plane cubic. (We’ll soon see that any plane cubic is a genus 1 curve.)

**Corollary.** $\mathcal{L}$ on $C$ is ample iff it has positive degree.
There is one more natural place to find line bundles: the normal bundle to a submanifold of dimension 1 less. I’ll now discuss normal bundles algebraically, and as an application prove the *adjunction formula*.

Suppose $D$ is a nonsingular divisor (codimension 1) on nonsingular $X$. Then we can understand the canonical sheaf of $D$ in terms of the canonical sheaf of $X$.

**The adjunction formula.** $\mathcal{K}_D = \mathcal{K}_X(D)|_D$.

(Remind them what $\mathcal{K}_X(D)$ means.)

This actually holds in much more generality, e.g. $D$ can be arbitrarily singular, and $X$ need not be smooth.

Here is an informal description of why this is true.

- **Motivation**: tangent space in differential geometry to a point $p$ in a manifold $W$ is the space of curves, modulo some equivalence relation.

- The cotangent space is the dual of this space, and can be interpreted as the space of functions vanishing at $p$ modulo an equivalence relation: the functions vanishing to order 2 at $p$. In the space $O$ of functions near $p$, the first is the ideal $I_p$, the second is the ideal $I_p^2$. Thus the cotangent space is $I_p/I_p^2$.

- **Aside**: in this case, $I_p$ is a maximal ideal $m_p$, as $O/I_p$ is a field. So the cotangent space is $m_p/m_p^2$. That’s why the Zariski tangent space is defined as $m_p/m_p^2$. This is a purely algebraic definition. It works for any algebraic $X$, not even defined over a field, not even non-singular.

- Next consider $Y$ to be a nonsingular subvariety of $X$, of codimension $c$. Then there is a conormal bundle of $Y$ in $X$ (the dual of the normal bundle. It has rank $d$. The conormal sheaf is the sheaf of sections of $Y$. It is locally free of rank $d$. **Definition (motivated by previous discussion)**. In sheaf language: the conormal bundle is $\mathcal{I}/\mathcal{I}^2$. This is a priori a sheaf on all of $X$, but in fact it lives on $Y$ (“is supported on $Y$”).

- Now suppose further that $Y$ is a divisor, so I’ll now call it $D$. Then $\mathcal{I} = O_X(D)$. We’re modding out this sheaf by functions vanishing on $D$; this is the same as restricting to $D$. Hence $\mathcal{I}/\mathcal{I}^2 \sim O(D)|_D$.

**Proof of the adjunction formula.** We have an exact sequence

$$0 \to T_D \to T_X|_D \to N_D \to 0.$$ 

Dualize to get:

$$0 \to N_D^* \to \Omega_X|_D \to \Omega_D \to 0.$$ 

Take top wedge powers to get $\mathcal{K}_X|_D \sim \mathcal{K}_D \otimes N_D^*$ from which

$$\mathcal{K}_D \sim \mathcal{K}_X|_D \otimes N_D = \mathcal{K}_X|_D \otimes O_X(D)|_D = \mathcal{K}_X(D)|_D.$$
3.1. **Applications of the adjunction formula.** 1) Cubics in \(\mathbb{P}^2\) have trivial canonical bundle. Hence all genus 1 curves have canonical sheaf that is not degree 0, but also trivial.

2) Quartics in \(\mathbb{P}^3\) also have trivial canonical bundle. K3 surfaces.

3) What’s the genus of a smooth degree \(d\) curve in \(\mathbb{P}^3\)? Answer: \((d-1)(d-2)/2\).

4) Smooth quartics in \(\mathbb{P}^2\) are embedded by their canonical sheaf. More on this in a second.

5) Smooth complete intersection of surfaces of degree 2 and 3 in \(\mathbb{P}^3\): also embedded by their canonical sheaf.

3.2. **Classification of curves of genus 3.** I’ll discuss at greater length next day.

**Theorem.** Every smooth curve of genus 3 is of (precisely) one of the two following forms.

(i) A smooth quartic curve in \(\mathbb{P}^2\) (in a unique way, up to automorphisms of \(\mathbb{P}^2\)).

(ii) A double cover of \(\mathbb{P}^1\) branched over 8 points (in a unique way, up to automorphisms of \(\mathbb{P}^2\)).