1. Castelnuovo’s Theorem

1.1. Motivation: Minimal rational surfaces

1.2. Motivation Luroth’s theorem (in characteristic 0)

2. Proof of Castelnuovo’s criterion (part 1)

On board beforehand:

- Useful trick. $|D| \neq \emptyset$ (i.e. $h^0(D) > 0$), $C$ irreducible, $C^2 \geq 0$ implies $DC \geq 0$.
- Genus formula. $2g(C) - 2 = C \cdot (K_S + C)$.
- Riemann-Roch: $\chi(D) = \chi(O) + \frac{1}{2}D \cdot (D - K)$.
- Riemann-Roch: In the case when $h^1(O) = 0$ and $h^0(K - D) = 0$, we have $h^0(D) \geq 1 + \frac{1}{2}D \cdot (D - K)$ (with equality iff $h^1(D) = h^2(O) = 0$).

1. CASTELNUOVO’S THEOREM

We saw how tricky it was to show that a surface is rational.

**Theorem: Castelnuovo’s Rationality Criterion.** Let $S$ be a surface with $q = P_2 = 0$. Then $S$ is rational.

Reminder. $q = h^1(S, O_S) = h^0(S, O_S) = h^2(S, O_S) = h^1(S, K_S)$ (draw Hodge diamond). This is called the **irregularity** of a surface.

$P_2 = h^0(S, K_S^{\otimes 2})$.

It was once believed that this could be weakened to $q = P_1 = 0$, which is somehow more attractive (as $P_1$ is an entry in the Hodge diamond), but this false, and we may see examples before the end of the course (Enriques surfaces, Godeaux surfaces).

1.1. Motivation: Minimal rational surfaces. We know lots of rational surfaces now: $\mathbb{P}^2$, $F_n$, and blow-ups of these. At this point, we may suspect that we’ve found them all. How can we show this? We’ll use Castelnuovo’s criterion.
1.2. **Motivation Lüroth’s theorem (in characteristic 0).** A variety \( V \) of dimension \( n \) is **unirational** if there is a dominant map (i.e. one with dense image) \( \mathbb{P}^n \rightarrow V \).

**Lüroth’s Theorem.** Every unirational curve is rational.

**Proof.** This is true in arbitrary characteristic, but here’s a proof that works only in characteristic 0. Suppose \( \mathbb{P}^1 \rightarrow C \), where \( C \) is a curve, possibly singular and not proper. Then we also get a rational map \( \mathbb{P}^1 \rightarrow C' \), where \( C' \) is a smooth compactification of a smoothing of \( C \). By our lemma from long ago, any rational map from a smooth curve to anything projective extends to a morphism, so we have \( \mathbb{P}^1 \rightarrow C' \). Dominant implies surjective. So we can apply the Riemann-Hurwitz formula, to see that

\[
2 - 2g(\mathbb{P}^1) = d(2 - 2g(C')) - \text{ramification contribution}.
\]

The left side is 2, but if \( g(C') > 0 \) the right side can’t be positive.

**Theorem.** In characteristic 0, every unirational surface is rational.

In positive characteristic, the theorem is **false!** Ask Ted Hwa for an example.

**Question:** where does the following argument break down in positive characteristic?

**Proof.** Suppose \( S \) is a unirational surface. If there was any doubt, let’s say that it is smooth and compact. (Otherwise, there is a way of producing a smooth and compact birational model.) So we have \( \mathbb{P}^2 \rightarrow S \). By the elimination of indeterminacy, we can blow up \( \mathbb{P}^2 \) and get a morphism \( \text{Bl}(\mathbb{P}^2) \rightarrow S \). This morphism is dominant and hence surjective. Interpret \( q(S) \) as \( H^0(S, \Omega_S) \), and recall \( P_2(S) = H^0(S, K_S^\otimes 2) \). If \( q > 0 \) or \( P_2 > 0 \), then pullback the nonzero form (i.e. section of either \( \Omega_S \) or \( K_S^\otimes 2 \)) to get a non-zero section of the corresponding bundle on \( \text{Bl}(\mathbb{P}^2) \). This would give \( q(\text{Bl}(\mathbb{P}^2)) > 0 \) or \( P_2(\text{Bl}(\mathbb{P}^2)) > 0 \).

Hence \( q(S) = P_2(S) = 0 \). Then by Castelnuovo, \( S \) is rational.

**Remark.** Even in characteristic 0, there are 3-folds that are unirational but not rational, and they are not even that exotic! It is not hard to show that smooth cubic threefolds in \( \mathbb{P}^4 \) are all unirational; Clemens and Griffiths showed that none of them are rational! Iskovskih and Manin did the same for quartic threefolds as well.

2. **Proof of Castelnuovo’s criterion (Part 1)**

We’ll make a couple of reduction steps.

**Castelnuovo.** Let \( S \) be a minimal surface with \( q = P_2 = 0 \). Then there exists a smooth rational curve \( C \) on \( S \) such that \( C^2 \geq 0 \). **Keep on board.**

**Proof that Castelnuovo implies Castelnuovo’s criterion.**

\( \mathcal{O}_S(C) \) clearly has a section, one whose zero set is \( C \). We’ll see that in fact \( h^0(S, \mathcal{O}_S(C)) \geq 2 \), so “the curve moves”. Consider \( 0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{O}_C(C) \rightarrow 0 \). Now \( q = h^1(S, \mathcal{O}_S) = \)
so when we take global sections, the sequence remains exact, so
\[
\begin{align*}
  h^0(S, \mathcal{O}_S(C)) &= h^0(S, \mathcal{O}_S) + h^0(C, \mathcal{O}_C(C)) \\
  &= 1 + C^2 - g(C) + 1 + h^1(C, \mathcal{O}_C(C)) \\
  &= 2 + C^2 \quad \text{(as } C \cong \mathbb{P}^1\text{, and } \mathcal{O}_C(C) \text{ has positive degree)} \\
  &\geq 2
\end{align*}
\]
So taking 2 sections, \( C \) and one other, we get a rational map \( S \dashrightarrow \mathbb{P}^1 \). After blowing up, this becomes a morphism \( \tilde{S} \dashrightarrow \mathbb{P}^1 \). One of its fibers is isomorphic to \( C \). By the Noether-Enriques theorem, it follows that \( S \) is rational. \( \square \)

So now we want to prove Castelnuovo’. Instead we’ll prove

**Castelnuovo”.** \( q = P_2 = 0 \) implies that there is an effective divisor \( E \) on \( S \) such that \( K \cdot E < 0 \) and \( |K + E| = 0 \). Keep on board: We seek \( |E| \neq |E + K| = 0, K \cdot E < 0 \).

**Castelnuovo” implies Castelnuovo’**. For then some component \( C \) of \( E \) satisfies \( K \cdot C < 0 \), and any component satisfies \( h^0(S, K + C) = 0 \). Applying Riemann-Roch to \( K + C \) we get
\[
\begin{align*}
  0 &= h^0(K + C) \\
  &\geq h^0(K + C) - h^1(K + C) + h^0(-C) \\
  &= \chi(K + C) \\
  &= \chi(\mathcal{O}_X) + \frac{1}{2}((K + C) - K) \cdot (K + C) \\
  &> h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) + \frac{1}{2}(C + K) \cdot C \\
  &\geq 1 + \frac{1}{2}(C + K) \cdot C \\
  &= g(C).
\end{align*}
\]
Hence \( g(C) = 0 \). \( (C + K) \cdot C = -2 \), hence \( C^2 \geq -1 \). If \( C^2 = -1 \), then \( C \) is an exceptional curve, and we hypothesized that there weren’t any. So Castelnuovo’ follows. \( \square \)

**Proof of Castelnuovo” in the case \( K^2 = 0 \).**

How can we possibly use \( P_2 = 0 \)? Only one reasonable way: Our hypothesis \( P_2 = 0 \) gives \( h^2(-K) = 0 \) (Serre duality). Hence by Riemann-Roch (and \( q = 0 \)):
\[
h^0(-K) \geq h^0(-K) - h^1(-K) + h^2(-K) = h^0(\mathcal{O}) - h^1(\mathcal{O}) + h^2(\mathcal{O}) + K^2 \geq 1 + K^2.
\]
(We’ll use this in the \( K^2 > 0 \) case too.)

So \( |-K| \neq 0 \). Let \( H \) be a hyperplane section of \( S \). Then \( H \cdot K < 0 \). Note:

- If \( n = 0 \), then \( |H + nK| \neq 0 \).
- If \( n \gg 0 \) then \( |H + nK| = 0 \) (as \( (H + nK) \cdot H < 0 \))
Thus there is an $n \geq 0$ such that $|H + nK| \neq \emptyset$, but $|H + (n + 1)K| = \emptyset$ as $|H| \neq \emptyset$, and $(H + nK) \cdot H < 0$ for $n \gg 0$. Let $D$ be an element. $|K + D| = \emptyset$, and $K \cdot D = -(K) \cdot H < 0$. \hfill \Box

**Proof of Castelnuovo” in the case $K^2 > 0$.**

Recall $h^0(−K) = 1 + K^2$, so $h^0(−K) \geq −2$. Suppose $D \in |−K|$. Three cases:

(1) There is a reducible choice of $D$, i.e. $A, B$ effective with $A + B \in |−K|$. Then $A \cdot K$ or $B \cdot K < 0$, say the former. Then $A$ is an effective divisor on $S$ such that $A \cdot K < 0$, and $|A + K| = |−B| = \emptyset$.

Case 1: There is a reducible choice of $D$, i.e. $A, B$ effective with $A + B \in |−K|$. Then $A \cdot K$ or $B \cdot K < 0$, say the former. Then $A$ is an effective divisor on $S$ such that $A \cdot K < 0$, and $|A + K| = |−B| = \emptyset$.

Case 2: $\text{Pic}(C) = ZK$. This is the only case where characteristic 0 comes up! From the exact sequence

$$H^1(S, \mathcal{O}_S) \rightarrow \text{Pic} S \rightarrow H^2(S, \mathbb{Z}) \rightarrow H^2(S, \mathcal{O}_S)$$

we have $H^2(S, \mathbb{Z}) \cong \text{Pic} S = ZK$. Thus $b_2 = 1$. By Poincare duality, the intersection form on $H^2(S, \mathbb{Z})$ is unimodular, so $K^2 = 1$. By Noether’s formula,

$$1 = \chi(\mathcal{O}_S) = \frac{1}{12}(K^2 + 2 - 2b_1 + b_2)$$

from which $b_1 = −4$, contradiction.

Case 3: All divisors $D$ in $|−K|$ irreducible and $\text{Pic}(C) \neq ZK$. Suppose $H$ were an effective divisor. As $|−K| \neq \emptyset$, there exists $n > 0$ such that $|H + nK| \neq \emptyset$ and $|H + (n + 1)K| = \emptyset$. If $(H + nK) \cdot K < 0$, we’d be done.

Take an $H$ such that $H + nK \neq 0$. Let $E \in |H + nK|$, $E = \sum n_iC_i$. Then $K \cdot E = −D \cdot E$, and by the useful remark $D \cdot E \geq 0$ since $D$ is irreducible. We are painfully close to being done: we have $K \cdot E \leq 0$, and we want $K \cdot E < 0$!

Thus $K \cdot C_i \leq 0$ for some $C = C_i$. Hence $|K + C| = \emptyset$, from which $0 = h^0(K + C) \geq 1 + \frac{1}{2}(C^2 + CK) = g(C)$. $g(C) = 0$, and $C^2 = −2 − K \cdot C$ (genus formula). We have gained exactly one thing in this paragraph: our divisor $C$ is irreducible, whereas our divisor $E$ was not necessarily. We know that $|C| \neq \emptyset$, $|K + C| = \emptyset$, and $K \cdot C \leq 0$, and we want to show that $K \cdot C < 0$.

So we’ll assume $K \cdot C = 0$, and find a contradiction. From the genus formula, $C^2 = −2$. We’ll calculate $h^0(−K − C)$. Note that $h^0(2K + C) = h^0(2K + (−D)) \leq h^0(K + C) = 0$. 

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Thus

\[ h^0(-K - C) \geq \chi(-K - C) = \chi(O_X) + \frac{1}{2}((K + C)^2 + K(K + C)) \]
\[ = 1 + \frac{1}{2}(C^2 + 3KC + 2K^2) \geq K^2 \geq 1 \]

Since \( C^2 = -2 \), we have \( C \neq -K \), so there exists a nonzero effective divisor \( A \) such that \( A + C \in |-K| \). This contradicts our hypothesis that \( |-K| \) has no reducible divisors.

All that’s left is:

**Proof of Castelnuovo** in the case \( K^2 < 0 \).