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Last day we looked at a lot of rational surfaces. We made use of:

**Useful proposition.** Consider the blow-up of $\mathbb{P}^2$ at $n$ general points, giving exceptional divisors $E_1, \ldots, E_n$. Then the intersection ring on $\mathbb{P}^2$ is given by

$$\mathbb{Z}[H, E_1, \ldots, E_n]/H^2 = 1, HE_i = 0, E_i \cdot E_j = 0, E_i^2 = -1.$$ 

We can understand divisors and sections of divisors in terms of divisors on $\mathbb{P}^2$ with certain multiplicities at the $E_j$. More precisely: the vector space of sections of $aH - b_1E_1 - \cdots - b_nE_n$ is naturally isomorphic to the vector space of degree $a$ polynomials in $\mathbb{P}^2$ vanishing with multiplicity at least $b_i$ on $E_i$.

We also proved the following. Suppose $S$ is a surface, and $-K_S$ gives a map to projective space. Then $(-1)$-curves map isomorphically onto lines. Conversely, if $S$ is a surface, and $-K_S$ gives a map to projective space, then any curve mapping isomorphically onto a line is a $(-1)$-curve.

Today: Cubic surfaces. But first, some interesting combinatorial remarks, due to Tyler, Diane and others.

1. Combinatorial aspects

**Proposition.** The automorphism group of the Peterson graph is $S_5$. (I haven’t drawn the Peterson graph in these notes, sorry!)

**Proof.** Two (equivalent) proofs. Diane’s: To each vertex put a size 2 subset of $\{1, \ldots, 5\}$. Join them by an edge if the don’t intersect.

*Date: Friday, November 15.*
Tyler’s: Label the edges as follows (3 have label 1, etc.). Observe that as soon as you’ve labeled one edge with a number, the other two are determined: they are those that are distance 3 from it. Hence if $G$ is the automorphism group of the graph, we have a morphism $G \to S_5$. It is surjective: you can get a 5-cycle (rotate) and a 2-cycle (do it). It is injective: suppose you have an automorphism fixing the colors. We’ll show that the three 1-labeled edges are fixed. Look at the two pairs of edges each 1-edged meets; note that this gives a partition of $\{2, 3, 4, 5\}$ into two couples. There are 3 ways to do this, and they correspond to the 3 edges.

Connection between Diane’s and Tyler’s: in Diane’s construction: label an edge with the number missing in its vertex labels.

If you remember, I described a pattern of automorphism groups of the intersection graphs of the blow-up of $\mathbb{P}^2$ at $n$ up to 9 points. The answer was:

$$
\begin{array}{cccccccc}
n = & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\dim \text{Aut } [S_3 \times S_2] & W(A_4) & W(D_5) & W(E_6) & W(E_7) & W(E_8) & W(E_9) \\
\end{array}
$$

I expressed disappointment that $S_3 \times S_2$ was not isomorphic to $D_{12}$, the symmetries of hexagon, because that’s what the correct answer was. But in fact:

**Proposition (Tyler).** $S_3 \times S_2 \cong D_{12}$.

**Proof.** $x = (12)(12)$ and $y = (12)(123)$ generate $S_3 \times S_2$. They also satisfy the relations defining $D_{12}$: $x^2 = y^6 = e$, $yx = y^{-1}$. □

**Remark.** The map $D_{12} \to S_3$ corresponds to the induced permutation of the 3 diagonals. The map $D_{12} \to S_2$ corresponds to the permutation of the two inscribed equilateral triangles.

## 2. del Pezzo surfaces

Another remark about these surfaces.

**Definition.** A surface $S$ is a del Pezzo surface if $-K_S$ is ample.

This is yet another ancient idea that remains important. Most recently, they have come up in physics.

Examples from last day: $\mathbb{P}^1 \times \mathbb{P}^1$, and the blow-up of $\mathbb{P}^2$ at up to 8 general points. Let’s get rid of that nasty word “general”, by making more precise which points you throw out.

**Proposition.** The blow up of $\mathbb{P}^2$ at up to 8 distinct points, no 3 on a line and no 6 on a conic, is a del Pezzo surface.

**Proof.** Check using our useful proposition that $-K$ or $-2K$ is very ample. If $n \leq 6$, $-K$ is ample, i.e. cubics vanishing at the $n$ points gives something that separates points.
and tangent vectors. If $n = 7$ or $8$, $-K$ isn’t very ample, but $-2K$ is: the linear system corresponding to sextics vanishing at the $n$ points separate points and tangent vectors.

**Proposition.** The blow up of $\mathbb{P}^2$ at up to 8 distinct points, with 3 on a line or 6 on a conic, is not a del Pezzo surface.

**Proof.** First, note that a del Pezzo surface can’t have a (-2)-curve, i.e. a genus 0 curve $C$ with self-intersection $\leq -2$. Reason: genus formula gives

$$-2 = 2g - 2 = C \cdot (K + C) = C \cdot K + C \cdot C \leq C \cdot K - 2$$

so $0 \leq C \cdot K$. But $-K$ is ample, so $K \cdot C < 0$ for all $C$.

If at least there are at least 3 points on a line, then there is a genus 0 curve of self-intersection at most $-2$.

Similarly, if there are at least 6 points on a conic, then there is a genus 0 curve of self-intersection at most $-2$.

**Remark: Slight extension.** You can allow “infinitely near” points: blow up $p_1$, and blow up a point $p_2$ on the exceptional divisor $E_1$; somewhat archaic (but still-used) terminology is that $p_2$ is an “infinitely near point” to $E_2$.

But as soon as you blow up a point on an exceptional divisor, you have a genus 0 curve of self-intersection at most -2, and hence it can’t be a del Pezzo surface.

**Theorem.** The only del Pezzo surfaces are the above blow-ups of $\mathbb{P}^2$ (up to 8 points, no 3 on a line, no 6 on a conic), and $\mathbb{P}^1 \times \mathbb{P}^1$.

**Proof:** later in the course.

3. **Cubic surfaces**

**Blow up $\mathbb{P}^2$ at six points.** 27 (-1)-curves. Reminder of where they are: 6 exceptional divisors $E_i$, $\binom{6}{2}$ lines $L_{ij}$, 6 conics $C_{ijklm}$.

Anticanonical map gives embedding in to $\mathbb{P}^3$. (Remember: $(-1)$-curves ↔ lines.)

Get smooth cubic surface in $\mathbb{P}^3$ with lines. We’ll see: (i) there are no more lines on this surface, (ii) that almost all smooth cubic surfaces are $\mathbb{P}^2$ blown up at six points and hence have 27 lines, (iii) all smooth cubic surfaces have 27 lines, and (iv) all smooth cubic surfaces are $\mathbb{P}^2$ blown up at six points.

**Proposition.** We have found all the lines.

**Proof.** Diophantine equations. Suppose $D = aH - \sum b_i E_i$ is a line, i.e. a $(-1)$-curve.
First case: some \( b_i < 0 \). Then \( D \cdot E_i = -b_i < 0 \). But \( D \) is the class of a line, so \( D \cdot E_i \geq 0 \) unless the line and \( E_i \) have a component in common. Thus they must be equal, i.e. \( D = E_i \). Hence we’ve found six lines: \( E_1, \ldots, E_6 \).

Second case: all \( b_i \geq 0 \). Also note that \( a > 0 \).

Then degree = 1 means \( (3H - \sum E_i) \cdot D = 1 \), from which \( 3a - \sum b_i = 1 \).

\( D^2 = -1 \) implies \( a^2 - \sum b_i^2 = -1 \).

Recall (or re-prove) the quadratic mean - arithmetic mean inequality:

\[
\sqrt{\frac{\sum b_i^2}{6}} \geq \frac{\sum b_i}{6}
\]

from which \( \sum b_i^2 \geq (\sum b_i)^2/6 \). Thus

\[
a^2 = \sum b_i^2 - 1 \geq (\sum b_i)^2/6 - 1 = (3a - 1)^2/6 - 1
\]

from which \( 6a^2 + 6 \geq 9a^2 - 6a + 1 \Rightarrow 0 \geq 3a^2 - 6a - 5 \Rightarrow 8 \geq 3(a - 1)^2 \). Thus \( a = 1 \) or 2.

If \( a = 1 \), then we are considering the class of lines, then we get \( \sum b_i = 2 \) and \( \sum b_i^2 = 2 \), from which two of the \( b_i \)'s are 1 and the rest are 0.

If \( a = 2 \), we get \( \sum b_i = 5 \) and \( \sum b_i^2 = 5 \). The five \( b_i \)'s must be equal to 1, and the last equal to 0. \( \square \)

**Proposition.** Almost all smooth cubic surfaces are \( \mathbb{P}^2 \) blown up at six points.

**Proof.** By dimension count. Dimension of space of cubic surfaces: count cubic equations in 4 variables (20). Subtract 1 to projectivize, to get 19. There is a \( \mathbb{P}^{19} \) parametrizing all smooth cubic surfaces. An open set \( S \) corresponds to the smooth ones.

Now how many ways can we get a cubic surface by blowing up six points in the plane? Choose six points in the plane (12 dimensions), except mod out by the automorphisms of the plane \( \dim PGL(3) = 8 \). Then map to projective space using four linearly independent sections of \( 3H - \sum E_i \). There is a four-dimensional space of sections, so we have 16 dimensions of choice of 4 sections. But any multiple of such a 4-tuple gives the same embedding, so \(-1\). Get:

\[
12 - 8 + 16 - 1 = 19.
\]

Thus we have a 19-dimensional family!

It’s not an open subset yet; there may be, and in fact are, many ways of representing the surface as a blow-up of six points. But there are only a finite number: any such description corresponds to six lines on the surface, no 2 intersecting. So the image of this 19-dimensional family lies in \( S \), and must be dense in \( S \). \( \square \)

We’ll see soon: Every smooth cubic has 27 lines, and is the blow-up of \( \mathbb{P}^2 \) at 6 points
3.1. **Automorphisms of the intersection graph.** Again, we can make a graph of the combinatorial structure of the 27 lines. It again is highly symmetric.

**Theorem.** Its automorphism group is $W(E_6)$, a finite group of order $51840 = 2^7 \times 3^4 \times 5$.

**Sketch of proof.** Recall the definition of $W(E_6)$. 6 skew lines gives full structure. Get map to $\mathbb{P}^2$. See $S_6$ in it. Last one: Cremona transformation.

Interpretation of this group in two ways:

Geometrically: as the monodromy of the 27 lines as you make loops in $S$.

Algebraically: Let $M$ be the variety parametrizing (surface, line). There is a morphism from $M$ to $S$, of degree 27. $M$ turns out to be irreducible. Hence we have a field extension of degree $27k(S) \subset k(M)$. If we take the Galois closure of this field extension, we get a field extension of order 51840, and its Galois group is $W(E_6)$.

Next day we’ll show that:

3.2. **Every smooth cubic has 27 lines, and is the blow-up of $\mathbb{P}^2$ at 6 points.** Let $S$ be a smooth cubic surface.

Earlier today (in response to a question of Eric’s) I said:

**Proposition.** $S$ is anticanonically embedded.

Proof: Adjunction formula. $S$ has degree 3, so $K_S = O_{\mathbb{P}^3}(-4 + 3)|_S$.

Hence lines correspond to $(-1)$-curves.

The strategy of proof that every smooth cubic is a blow-up of $\mathbb{P}^2$ at 6 points is as follows.

**Proposition.** Every cubic surface (even singular ones) contains a line.

**Proposition.** Given any line $l$ in $S$, there are exactly 10 other lines in $S$ meeting $l$ (distinct from $l$). These fall into 5 disjoint pairs of concurrent lines.

**Corollary.** Any smooth cubic contains two skew lines. Finally:

**Theorem.** $S$ is the blow-up of $\mathbb{P}^2$ at 6 points.

We’ll see instead that $S$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 5 points. We’d seen earlier that $\mathbb{P}^2$ blown up at 2 points is $\mathbb{P}^1$ blown up at one point, so we’ll be done.

**Proof.** Let $l$ and $l'$ be disjoint lines. We define rational maps $\phi : l \times l' \dashrightarrow S$ and $\psi : S \dashrightarrow l \times L'$ as follows: if $(p, p')$ is a generic point of $l \times l'$, the line $\langle p, p' \rangle$ meets $S$ in a third point $p''$. Define $\phi(p, p') = p''$. For $s \in S - l - l'$, set $p = l \cap \langle s, l' \rangle$, $p' = l' \cap \langle s, l \rangle$ and put $\phi(s) = (p, p')$. It is clear that $\phi$ and $\psi$ are inverses. Moreover, $\phi$ is a morphism: we can define at a points of $l$ (or $l'$) by replacing the plane $\langle s, l \rangle$ by the tangent plane to $S$ at $s$. CG
(checking that this gives a morphism, explain). Thus $\psi$ is a birational morphism, and is a composite of blow-ups. Which curves are blown-down? Those lines meeting both $l$ and $l'$. We’ll see why there are precisely 5 of them, one of each the pairs described above. □