1. Fun with rational surfaces

I’ll start again, but skip over things we did last time.

**The surface** $\mathbb{P}_0$. This is $\mathbb{P}^1 \times \mathbb{P}^1$. Its intersection theory is $\mathbb{Z}[h, f]/h^2 = f^2 = 0$, $hf = 1$. $h$ and $f$ are the classes of fibers of the two projections to $\mathbb{P}^1$. These are traditionally called *rulings*. Using the divisor $h + f$, we can embed $\mathbb{P}_0$ as a smooth quadric in $\mathbb{P}^3$. More precisely than last day: Let $([x; y], [u; v])$ be co-ordinates to $\mathbb{P}^1 \times \mathbb{P}^1$. Then $xu, xv, yu, yv$ are sections of $\mathcal{O}(h + f)$, that separate points and tangent vectors, and hence give a closed immersion into $\mathbb{P}^3$. If the base field is algebraically closed, all smooth quadrics are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Now, what do the rulings look like in projective space? What’s their degree? Answer: $h \cdot (h + f) = 1$. They are lines! (Draw pictures of a hyperboloid with one sheet, and show them the lines.) Question: where are the lines in an ellipsoid? Answer: they are complex, so don’t be misled by the real picture.

Many of the other surfaces corresponds to blow-ups of $\mathbb{P}^2$ at a certain number of points. Before discussing them, here is a useful proposition that I mentioned last day.

**Useful proposition.** Consider the blow-up of $\mathbb{P}^2$ at $n$ general points, giving exceptional divisors $E_1, \ldots, E_n$. Then the intersection ring on $\mathbb{P}^2$ is given by

$$\mathbb{Z}[H, E_1, \ldots, E_n]/H^2 = 1, HE_i = 0, E_i \cdot E_j = 0, E_i^2 = -1.$$ 

We can understand divisors and sections of divisors in terms of divisors on $\mathbb{P}^2$ with certain multiplicities at the $E_j$. More precisely: the vector space of sections of $aH - b_1E_1 - \cdots - b_nE_n$ is naturally isomorphic to the vector space of degree $a$ polynomials in $\mathbb{P}^2$ vanishing with multiplicity at least $b_i$ on $E_i$.

*Date: Wednesday, November 13.*
Sketch of proof. Any divisor in \( O(aH - b_1E_1 - \cdots - b_nE_n) \) pushes forward to some divisor \( D \) in class \( aH \) on \( \mathbb{P}^2 \). The strict transform of that divisor (i.e., somewhat hideously: the closure of its preimage where the blow-up is an isomorphism) is in class \( aH - (\text{mult}_{p_i} D)E_1 - \cdots - (\text{mult}_{p_n} D)E_n \). So the actually original divisor must be this, plus some more \( E_i \)'s, from which \( b_i \leq \text{mult}_{p_i} D \).

The vector space structure is the same, as both are subvector spaces of the sections over \( \mathbb{P}^2 - p_1 - \cdots - p_n \).

The surface \( \mathbb{F}_1 \). As observed before, \( \mathbb{F}_1 \) is (isomorphic to) the blow-up of \( \mathbb{P}^2 \) at a point.

Consider the divisor class \( 2H - E_1 \). This corresponds to conics in \( \mathbb{P}^2 \) through \( p_1 \), which gives a five-dimensional vector space. It separates points and tangent vectors (somewhat obviously away from \( E_1 \), and not so obviously along \( E_1 \)). Thus we get an immersion of \( \mathbb{F}_1 \) into \( \mathbb{P}^4 \). Its degree is \((2H - E) \cdot (2H - E) = 3 \). We get a cubic surface in \( \mathbb{P}^4 \).

Interpretation as projection from Veronese surface. Recall that we had an embedding of \( \mathbb{P}^2 \) into \( \mathbb{P}^5 \) via all conics. We can interpret \( \mathbb{F}_1 \) and its cubic embedding in \( \mathbb{P}^4 \) as a projection as follows. Suppose \( \mathbb{F}_1 = \text{Bl}_{[0;0;1]} \mathbb{P}^2 \).

Project the quartic Veronese surface in \( \mathbb{P}^5 \) down into \( \mathbb{P}^4 \) from \([0; 0; 0; 0; 0; 1]\). This is well defined except at that one point of the Veronese surface. We have a rational map \( \mathbb{P}^2 \dashrightarrow \mathbb{P}^4 \). By the elimination of indeterminacy theorem, we can resolve the map after some blow-ups of \( \mathbb{P}^2 \). In fact, we need only one: \( \mathbb{F}_1 \to \mathbb{P}^2 \).

Blow up \( \mathbb{P}^2 \) at two points. Now blow up \( \mathbb{P}^2 \) at two points. Where are the \((-1)\)-curves? There are obviously two: our two exceptional divisors. But there is one more: the proper transform of the line joining the two points. It has self-intersection number \((H - E_1 - E_2)^2 = -1 \).

When you blow down that “bonus” rational curve, what do you get? In fact: \( \mathbb{P}^1 \times \mathbb{P}^1 \). Last time, Eric saw this by interpreting what we just did as an elementary transformation of \( \mathbb{F}_1 \).

Here’s another way of seeing it. The divisor \( 2H - E_1 - E_2 \) corresponds to conic through our points, and is very ample. It gives an immersion into \( \mathbb{P}^3 \), and it is degree 2. Get smooth quadric surface. But we know that all smooth quadrics are \( \mathbb{P}^1 \times \mathbb{P}^1 \).

What are the maps to the two \( \mathbb{P}^1 \)'s? Answer: projection from each of two points. You can see why the proper transform of the line gets sent to a point under these two projections.
Blow up \(\mathbb{P}^2\) at three points, no two on a line. Where are the \((-1)\)-curves? Answer: the 3 exceptional divisors \(E_1, E_2, E_3\), but also the (proper transforms of) the lines through pairs of those points \(H - E_i - E_j\).

We can make a diagram showing how these six curves intersect, with a vertex for each curve, and an edge for each intersection. In this case, we get a hexagon.

If you blow down those other 3 curves? Answer: \(\mathbb{P}^2\). The rational map \(\mathbb{P}^2 \rightarrow \mathbb{P}^2\) is ancient, and is called a Cremona transformation.

Description 1: conics through these 3 points, i.e. \(2H - E_1 - E_2 - E_3\). The space of all conics is a six-dimensional vector space, and if we require the conics to vanish at the three points, we knock the vector space down to dimension 3. So this gives a map to \(\mathbb{P}^2\). It blows down the line \(H - E_1 - E_2\).

Description 2: Consider the rational map \(\mathbb{P}^2 \rightarrow \mathbb{P}^2\) given by \([x; y; z] \mapsto [1/x; 1/y; 1/z]\). By the elimination of indeterminacy theorem, we can resolve this map after some blow-ups. In fact, we need 3.

For fun, let’s look more closely. If \(x = 0\) and \(y, z \neq 0\), then we map to \([1; 0; 0]\); that’s a line being blown down. If \(x = 0\) and \(y = 0\) and \(z \neq 0\), it isn’t so clear. In fact, if \([x; y; z] = [at, bt, 1]\) where \(a\) and \(b\) aren’t both 0, then \([x; y; z] \mapsto [1/a; 1/b; t]\). Then let \(t\) go to 0, and we see that the limit is \([1/a; 1/b; 0]\). Thus the limit depends on the path of approach, and this is a point where the rational map is undefined, and will have to be blown up. That’s \(p_4\). And so on...

Aside: What happens if you blow up at 3 points on a line? Get a (-2)-curve.

Blow up \(\mathbb{P}^2\) at four points. Blow up at 4 general points. Get \(10 = 4 + \binom{4}{3}\) (-1)-curves.

Make graph. This is a highly symmetric graph, and is called the Peterson graph. Its automorphism group is \(S_5\); both Tyler and Diane gave me arguments for this within minutes of the end of class.

Blow up \(\mathbb{P}^2\) at five points. Blow up at 5 general points. Get \(5 + \binom{5}{2} = 15\), plus one more: conic. Cubics through these points. Get quartic surface in \(\mathbb{P}^4\). (It is the complete intersection of two complete quadrics.

The (-1)-curves turn into lines: \(E_1 \cdot (3H - \sum E_i) = 1. (H - E_1 - E_2) \cdot (3H - \sum E_i) = 1. (2H - \sum E_i) \cdot (3H - \sum E_i) = 1.\)

We could have saved ourselves some effort by instead noting that \(K_S = -3H + \sum E_i\). The divisor we are using to embed is \(-K_S\). If \(E\) is a (-1)-curve, then we know that \(E^2 = -1\), and by the genus formula \((K_S + E) \cdot E = -2\), from which \((-K_S \cdot E) = 1\). Note that we can reverse this: if \(S\) is embedded by the anticanonical divisor, then lines correspond to (-1)-curves. Exercise.

Blow up \(\mathbb{P}^2\) at six points. Now this is serious. Blow up at 6 points: Get \(6 + \binom{6}{2} + \binom{6}{3} = 6 + 15 + 6 = 27\).
Get smooth cubic surface in $\mathbb{P}^3$! With 27 lines! We’ll prove (next day) (i) there are no more lines on this surface, (ii) that almost all smooth cubic surfaces are $\mathbb{P}^2$ blown up at six points and hence have 27 lines, (iii) all smooth cubic surfaces have 27 lines, and (iv) all smooth cubic surfaces are $\mathbb{P}^2$ blown up at six points.

The automorphism group of the graph of exceptional curves is $W(E_6)$, the Weyl group of $E_6$.

**Blow up $\mathbb{P}^2$ at seven, eight, nine points.**

Get 56, [I can’t remember the number, maybe 148], $\infty$ lines. More interesting geometry here too. For example, blow up at 7 points. Get map to $\mathbb{P}^2$. It is a double cover. $(-1)$-curves map to lines. Fact: the branch locus is a quartic plane curve. $(-1)$-curves map 2-to-1 to the 28 bitangents of a smooth quartic plane curve.

The automorphism group of the graph of exceptional curves is $W(E_n)$, for $n = 7, 8, 9$. 