MODERN ALGEBRA (MATH 210) PROBLEM SET 7

1. Prove that \( \mathbb{Q}(\pi) \cong \mathbb{Q}(x) \). You may use the fact that \( \pi \) is transcendental.

2. Suppose \( q \) is a prime power. Show that there are \((q^3 - q)/3\) irreducible monic degree 3 polynomials in \( \mathbb{F}_q[x] \). (Hint: Consider the elements of \( \mathbb{F}_q^*, \) and their minimal polynomials over \( q \).) How many irreducible monic degree 12 polynomials are there in \( \mathbb{F}_q[x] \)?

3. Let \( E \) be the field \( k(x) \). Consider the six automorphisms of \( E \) given by mapping \( f(x) \) to \( f(x), f(1-x), f(1/x), f(1-1/x), f((1/1-x)), f(x/(x-1)) \) respectively. Show that these automorphisms form a group. Let \( F \) be the fixed point field. Show that \( I = (x^2 - x + 1)^3/x^3(x-1)^2 \in F \). Show that \( F = k(I) \) and \( [E : F] = 6 \). (Hint: Find a sixth degree equation with coefficients in \( k(I) \) satisfied by \( x \). Use a useful theorem from class.) Hence \( k(x)/k(I) \) is a Galois extension. What is its Galois group?

4. Suppose \( p_1, \ldots, p_n \) are distinct prime numbers, and let \( F = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n}) \). Find the degree of the extension \( F/\mathbb{Q} \). Show that \( F/\mathbb{Q} \) is Galois, and find its Galois group.

5. Suppose \( e_1, \ldots, e_n \) are the elementary symmetric functions in the \( n \) variables \( x_1, \ldots, x_n \) over some field \( k \), i.e. \( e_1 = x_1 + \cdots + x_n, e_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n, \ldots, e_n = x_1x_2 \cdots x_n \). Let \( p_i = x_i^a + \cdots + x_i^n \) (\( a \) a positive integer). By the theorem of symmetric functions, \( p_i \) is a polynomial in the \( e_j \). Show this explicitly as follows. We have seen that the polynomial \( f(T) \in (k(x_1, \ldots, x_n))[T] \)

\[
f(T) = T^n - e_1 T^{n-1} + e_2 T^{n-2} - \cdots + (-1)^{n-1} e_{n-1} T + (-1)^n e_n
\]

has roots \( x_1, \ldots, x_n \). Rewrite \( x_j^a f(x_1) + \cdots + x_j^a f(x_n) = 0 \) in terms of \( p_i, \ldots, p_{a+n} \) (\( a \) a positive integer), and show that if \( p_i \) is a polynomial in the \( e_j \) for \( i < a + n \), then \( p_{a+n} \) is too. Deal also with the cases where \( 1 \leq i \leq n \).

6. Suppose \( E/F \) is a Galois extension of degree \( n \) with Galois group \( \{\sigma_1, \ldots, \sigma_n\} \), and \( a_1, \ldots, a_n \) is a basis for \( E \) over \( F \). Show that

\[
\det \begin{pmatrix}
\sigma_1(a_1) & \sigma_1(a_2) & \cdots & \sigma_1(a_n) \\
\sigma_2(a_1) & \sigma_2(a_2) & \cdots & \sigma_2(a_n) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_n(a_1) & \sigma_n(a_2) & \cdots & \sigma_n(a_n)
\end{pmatrix}^2
\]

is a non-zero element of \( F \). (Hint: Show that it is non-zero separately. Recall the proof that if \( \{\sigma_1, \ldots, \sigma_n\} \) is a group of automorphisms of a field \( E' \), and \( F' \) is the fixed field, then \( [E' : F'] = n \). The proof used the fact that there could be no nontrivial solution to a certain linear equation.)

The set is due Tuesday, December 3 at 3:30 pm in Pierre Albin’s mailbox.

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Date: Tuesday, November 26, 2002.

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