MODERN ALGEBRA (MATH 210) PROBLEM SET 6

1. If $q$ is prime, show that the automorphism group of the group $\mathbb{Z}/q$ is isomorphic to $\mathbb{Z}/(q-1)$.

2. Suppose $p$ and $q$ are primes such that $p|q-1$. Show the existence of a nonabelian group of order $pq$. (Hint: try a semi-direct product.)

3. If $p$ is prime, prove that every group $G$ of order $2p$ is either cyclic or isomorphic to $D_{2p}$. 
   Hint: By Cauchy’s theorem, $G$ must contain an element of order $p$, and $\langle a \rangle$ is normal in $G$ because it has index 2.

   Problem 2 is part of the classification of groups of order $pq$, where $p < q$ are prime. The rest is given by: (i) If $p$ doesn’t divide $q - 1$, then all groups of order $pq$ are cyclic. (ii) If $p$ divides $q - 1$, there is only one nonabelian group of order $pq$. Problem 3 is a special case of (ii).

4. Suppose $G$ is a finite group of order $mn$, $\gcd(m, n) = 1$. Suppose $K$ is a normal subgroup of order $m$. Then $K$ has a complement iff $G$ has a subgroup of order $n$.

5. (Using semidirect products to see interesting phenomena.) Suppose $G$ is a finite group, and you want to show that $g$ normalizes $A \leq G$, i.e. that $g^{-1}Ag = A$. Then it suffices to show that $g^{-1}Ag \subseteq A$. Show that this isn’t true for infinite groups (i.e. that it is possible to have $g^{-1}Ag \not\subseteq A$) as follows. Let $K = \mathbb{Q}$, and $Q = \mathbb{Z} = \langle x \rangle$. Let $A \subseteq K$ be given by $Z \subseteq \mathbb{Q}$. Choose a $\phi: Q \to \text{Aut} K$ so that if $G = K \rtimes_{\phi} Q, xA^{-1} \not\subseteq A$.

6. Show that the splitting field of $x^3 - 3x + 1$ is degree 3 over $\mathbb{Q}$. (Hint: If $\alpha$ is a root, show that $1/(1 - \alpha)$ is also a root.)

7. Show that if an irreducible cubic in $\mathbb{Q}[x]$ has two complex roots and one real root, then its splitting field is a degree 6 extension of $\mathbb{Q}$.

8. (In preparation for showing the insolvability of the quintic.) Prove that if $\sigma$ is a 5-cycle and $\tau$ is a transposition in $S_5$, then $S_5$ is generated by $\{\sigma, \tau\}$.

The set is due Tuesday, November 26 at 3:30 pm in Pierre Albin’s mailbox.

Date: Tuesday, November 19, 2002.