MODERN ALGEBRA (MATH 210) PROBLEM SET 2

From the text: 2.69, 2.76, 2.98, 2.103. (For the last, and for future reference, read and understand the proof of Burnside’s Lemma, Theorem 2.113. Caution: the answer given in 2.103(ii) is incorrect!)

A1. This exercise shows that for \( n \neq 6 \), every automorphism of \( S_n \) is inner. Fix an integer \( n \geq 2 \) with \( n \neq 6 \).

(a) Prove that the automorphism group of a group \( G \) permutes the conjugacy classes of \( G \), i.e. for each \( \sigma \in \text{Aut}(G) \) and each conjugacy class \( \mathcal{K} \) of \( G \) the set \( \sigma(\mathcal{K}) \) is also a conjugacy class of \( G \).

(b) Let \( \mathcal{K} \) be the conjugacy class of transpositions in \( S_n \) and let \( \mathcal{K}' \) be the conjugacy class of any element of order 2 in \( S_n \) that is not a transposition. Prove that \( |\mathcal{K}| \neq |\mathcal{K}'| \). Deduce that any automorphism of \( S_n \) sends transpositions to transpositions.

(c) Prove that for each \( \sigma \in \text{Aut}(S_n) \)

\[ \sigma : (12) \mapsto (ab_2), \quad \sigma : (13) \mapsto (ab_3), \ldots, \sigma : (1n) \mapsto (ab_n) \]

for some distinct integers \( a, b_2, b_3, \ldots, b_n \in \{1, 2, \ldots, n\} \).

(d) Show that \((12), (13), \ldots, (1n)\) generate \( S_n \) and deduce that any automorphism of \( S_n \) is uniquely determined by its action on these elements. Use (c) to show that \( S_n \) has at most \( n! \) automorphisms and conclude that \( \text{Aut}(S_n) = \text{Inn}(S_n) \) for \( n \neq 6 \).

A2. We now show that \( \text{Inn}(S_6) \) is of index at most 2 in \( \text{Aut}(S_6) \). Let \( \mathcal{K} \) be the conjugacy class of transpositions in \( S_6 \) and let \( \mathcal{K}' \) be the conjugacy class of any element of order 2 in \( S_6 \) that is not a transposition. Prove that \( |\mathcal{K}| \neq |\mathcal{K}'| \) unless \( \mathcal{K}' \) is the conjugacy class of products of three disjoint transpositions. Deduce that \( \text{Aut}(S_6) \) has a subgroup of index at most 2 which sends transpositions to transpositions. Then prove that \( |\text{Aut}(S_6) : \text{Inn}(S_6)| \leq 2 \).

A3. Finally, we exhibit an outer automorphism of \( S_6 \). (There are other, more beautiful, descriptions.) Let \( t_1 = (12)(34)(56), t_2 = (14)(25)(36), t_3 = (13)(24)(56), t_4 = (12)(36)(45), t_5 = (14)(23)(56) \).

Show that \( t_1, \ldots, t_5 \) satisfy the following relations:

\[ (t_i^2) = e \text{ for all } i; \]
\[ (t_i^2 t_j^2) = e \text{ for all } i \text{ and } j \text{ with } |i - j| \geq 2; \]
\[ (t_i^3 t_j^3) = e \text{ for all } i \text{ and } j \text{ with } |i - j| = 1. \]

Use this to show that the map \((i(i+1)) \mapsto t_i^i \) gives an automorphism of \( S_6 \). (In the process, you will likely have to show that the relations above define \( S_6 \). Your argument will also presumably prove the obvious generalization to \( S_n \).)

Date: Tuesday, October 15, 2002.
B1. If there exists a chain of subgroups \( G_1 \leq G_2 \leq \cdots \leq G \) such that \( G = \bigcup_{i=1}^{\infty} G_i \) and each \( G_i \) is simple, then \( G \) is simple. (Note that \( G \) need not be finite!)

B2. (a) Let \( \Omega \) be an infinite set. Let \( D \) the subgroup of \( S_\Omega \) consisting of permutations which move only a finite number of elements of \( \Omega \) and let \( A \) be the set of all elements \( \sigma \in D \) such that \( \sigma \) acts as an even permutation on the (finite) set of points it moves. Prove that \( A \) is an infinite simple group.

(b) Prove that if \( H \neq \{e\} \) is a normal subgroup of \( S_\Omega \), then \( H \) contains \( A \), i.e. \( A \) is the unique nontrivial minimal normal subgroup of \( S_\Omega \).

C. For any finite group \( P \), let \( d(P) \) be the minimum number of generators of \( P \) (so, for example, \( d(P) = 1 \) iff \( P \) is a nontrivial cyclic group). Let \( m(P) \) be the maximum of the integers \( d(A) \) as \( A \) runs over all abelian subgroups of \( P \). Define

\[
J(P) = \langle A : \text{\( A \) is an abelian subgroup of \( P \) with \( d(A) = m(P) \)} \rangle.
\]

(\( J(P) \) is called the Thompson subgroup of \( P \). It plays a pivotal role in the study of finite groups, and in particular the classification of finite simple groups.)

(a) Prove that \( J(P) \) is preserved by all automorphisms of \( P \). (This is the definition of a characteristic subgroup.) Hence show that \( J(P) \) is normal.

(b) For both \( P = Q_8 \) and \( D_8 \), list all abelian subgroups \( A \) of \( P \) that satisfy \( d(A) = m(P) \). In both cases show that \( J(P) = P \).

(c) Prove that if \( Q \leq P \) and \( J(P) \) is a subgroup of \( Q \), then \( J(P) = J(Q) \). Deduce that if \( P \) is a subgroup (not necessarily normal) of the finite group \( G \) and \( J(P) \) is contained in some subgroup \( Q \) of \( P \) such that \( Q \) is normal in \( G \), then \( J(P) \) is normal in \( G \) as well.

Problems A1–C are from Dummit and Foote.

The set is due Tuesday, October 22 at 3:30 pm in Pierre Albin’s mailbox (opposite the elevator on the first floor of Building 380).