MODERN ALGEBRA (MATH 210) PROBLEM SET 2

From the text: 2.69, 2.76, 2.98, 2.103. (For the last, and for future reference, read and understand the proof of Burnside’s Lemma, Theorem 2.113. Caution: the answer given in 2.103(ii) is incorrect!)

A1. This exercise shows that for \( n \neq 6 \), every automorphism of \( S_n \) is inner. Fix an integer \( n \geq 2 \) with \( n \neq 6 \).

(a) Prove that the automorphism group of a group \( G \) permutes the conjugacy classes of \( G \), i.e. for each \( \sigma \in \text{Aut}(G) \) and each conjugacy class \( \mathcal{K} \) of \( G \) the set \( \sigma(\mathcal{K}) \) is also a conjugacy class of \( G \).

(b) Let \( \mathcal{K} \) be the conjugacy class of transpositions in \( S_n \) and let \( \mathcal{K}' \) be the conjugacy class of any element of order 2 in \( S_n \) that is not a transposition. Prove that \( |\mathcal{K}| \neq |\mathcal{K}'| \). Deduce that any automorphism of \( S_n \) sends transpositions to transpositions.

(c) Prove that for each \( \sigma \in \text{Aut}(S_n) \)

\[
\sigma : (12) \mapsto (ab_2), \quad \sigma : (13) \mapsto (ab_3), \ldots, \sigma : (1n) \mapsto (ab_n)
\]

for some distinct integers \( a, b_2, b_3, \ldots, b_n \in \{1, 2, \ldots, n\} \).

(d) Show that \((12),(13), \ldots, (1n)\) generate \( S_n \) and deduce that any automorphism of \( S_n \) is uniquely determined by its action on these elements. Use (c) to show that \( S_n \) has at most \( n! \) automorphisms and conclude that \( \text{Aut}(S_n) = \text{Inn}(S_n) \) for \( n \neq 6 \).

A2. We now show that \( \text{Inn}(S_6) \) is of index at most 2 in \( \text{Aut}(S_6) \). Let \( \mathcal{K} \) be the conjugacy class of transpositions in \( S_6 \) and let \( \mathcal{K}' \) be the conjugacy class of any element of order 2 in \( S_6 \) that is not a transposition. Prove that \( |\mathcal{K}| \neq |\mathcal{K}'| \) unless \( \mathcal{K}' \) is the conjugacy class of products of three disjoint transpositions. Deduce that \( \text{Aut}(S_6) \) has a subgroup of index at most 2 which sends transpositions to transpositions. Then prove that \( |\text{Aut}(S_6) : \text{Inn}(S_6)| \leq 2 \).

A3. Finally, we exhibit an outer automorphism of \( S_6 \). (There are other, more beautiful, descriptions.) Let \( t'_1 = (12)(34)(56), t'_2 = (14)(25)(36), t'_3 = (13)(24)(56), t'_4 = (12)(36)(45), t'_5 = (14)(23)(56) \). Show that \( t'_1, \ldots, t'_5 \) satisfy the following relations:

- \( (t'_i)^2 = e \) for all \( i \);
- \( (t'_i t'_j)^2 = e \) for all \( i \) and \( j \) with \( |i - j| \geq 2 \);
- \( (t'_i t'_j)^3 = e \) for all \( i \) and \( j \) with \( |i - j| = 1 \).

Use this to show that the map \( (i(i+1)) \mapsto t'_i \) gives an automorphism of \( S_6 \). (In the process, you will likely have to show that the relations above define \( S_6 \). Your argument will also presumably prove the obvious generalization to \( S_n \).)

Date: Tuesday, October 15, 2002.
B1. If there exists a chain of subgroups $G_1 \leq G_2 \leq \cdots \leq G$ such that $G = \bigcup_{i=1}^{\infty} G_i$ and each $G_i$ is simple, then $G$ is simple. (Note that $G$ need not be finite!)

B2. (a) Let $\Omega$ be an infinite set. Let $D$ the subgroup of $S_{\Omega}$ consisting of permutations which move only a finite number of elements of $\Omega$ and let $A$ be the set of all elements $\sigma \in D$ such that $\sigma$ acts as an even permutation on the (finite) set of points it moves. Prove that $A$ is an infinite simple group.

(b) Prove that if $H \neq \{e\}$ is a normal subgroup of $S_{\Omega}$, then $H$ contains $A$, i.e. $A$ is the unique nontrivial minimal normal subgroup of $S_{\Omega}$.

C. For any finite group $P$, let $d(P)$ be the minimum number of generators of $P$ (so, for example, $d(P) = 1$ iff $P$ is a nontrivial cyclic group). Let $m(P)$ be the maximum of the integers $d(A)$ as $A$ runs over all abelian subgroups of $P$. Define

$$J(P) = \langle A : A \text{ is an abelian subgroup of } P \text{ with } d(A) = m(P) \rangle.$$  

($J(P)$ is called the Thompson subgroup of $P$. It plays a pivotal role in the study of finite groups, and in particular the classification of finite simple groups.)

(a) Prove that $J(P)$ is preserved by all automorphisms of $P$. (This is the definition of a characteristic subgroup.) Hence show that $J(P)$ is normal.

(b) For both $P = Q_8$ and $D_8$, list all abelian subgroups $A$ of $P$ that satisfy $d(A) = m(P)$. In both cases show that $J(P) = P$.

(c) Prove that if $Q \leq P$ and $J(P)$ is a subgroup of $Q$, then $J(P) = J(Q)$. Deduce that if $P$ is a subgroup (not necessarily normal) of the finite group $G$ and $J(P)$ is contained in some subgroup $Q$ of $P$ such that $Q$ is normal in $G$, then $J(P)$ is normal in $G$ as well.

Problems A1–C are from Dummit and Foote.

The set is due Tuesday, October 22 at 3:30 pm in Pierre Albin’s mailbox (opposite the elevator on the first floor of Building 380).