All rings are commutative rings with 1. The problems are not in order of increasing difficulty. The problems have equal value. If you have any questions, just ask.

P1.
(a) Show that if $G$ contains a subgroup $H$ of index $m$, then it contains a normal subgroup $K$ lying in $H$ such that $|G : K|$ is finite and divides $m$.
(b) Show that when $n \neq 4$, the only proper subgroup of index less than $n$ in $S_n$ is the alternating group $A_n$.

P2. Suppose a finite group $G$ acts on a finite set $S$. Show that the average number of fixed points (over the elements of $G$) is the number of orbits. (As an application, on average the automorphisms of the icosahedron fix precisely one vertex.)

P3. Prove that an ideal $I$ of a ring $R$ is a maximal ideal if and only if $R/I$ is a field.

P4. We have shown that $x^2 + y^2 - 1$ is irreducible in $\mathbb{Q}[x]$. Classify all fields $F$ such that $x^2 + y^2 - 1$ is reducible in $F[x]$. (Note that $x^2 + y^2 - 1$ makes sense in $F[x]$ for any field $F$.)

P5. Let $G$ be a finite abelian group of odd order, written multiplicatively. Prove that every $x \in G$ has a unique square root; that is, there exists exactly one $g \in G$ with $g^2 = x$.

P6.
(a) Let $G$ be an arbitrary, possibly nonabelian, group, and let $S$ and $T$ be normal subgroups of $G$. Prove that if $S \cap T = \{1\}$, then $st = ts$ for all $s \in S$ and $t \in T$. (Hint: show that $sts^{-1}t^{-1} \in S \cap T$.)
(b) Suppose $G = S_1S_2\cdots S_n$, where the $S_i$ are normal subgroups; that is, for each $a \in G$, there are $s_i \in S_i$ for all $i$, with

$$a = s_1s_2\cdots s_n.$$ 

Then the following conditions are equivalent.
(i) $G = S_1 \times S_2 \times \cdots \times S_n$.
(ii) Every $a \in G$ has a unique expression of the form $a = s_1s_2\cdots s_n$, where $s_i \in S_i$ for all $i$.
(iii) For each $i$, $S_i \cap (S_1S_2\cdots S_{i-1}S_{i+1}\cdots S_n) = \{e\}$.