Instructions. Print your name and student ID number and write your signature to indicate that you accept the honor code. There are eight two-sided pages including this one, and twelve questions. Before you begin the exam, please make sure that you have all the pages. Read each question carefully and, unless specified otherwise, show all your work and explain your answers. The exam is open book and open note, but calculators are not permitted. You have 3 hours to complete the exam.
1. (20 points) Compute the inverse of

\[
A = \begin{pmatrix}
1 & 2 & 3 & 0 \\
0 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

in two different ways.
2. (15 points) Compute the determinant of

\[
A = \begin{pmatrix}
0 & 2 & 3 & 4 & 5 \\
2 & 2 & 2 & 2 & 2 \\
2 & 4 & 4 & 2 & 2 \\
1 & 3 & 4 & 3 & 5 \\
1 & 1 & 1 & 3 & 3 \\
\end{pmatrix}.
\]
3. TRUE or FALSE. Suppose $A$ and $B$ are orthogonal matrices.

(a) (3 points) $ABA^{-1}$ is orthogonal.
(b) (3 points) $A^{-1} = A^T$.
(c) (3 points) $A + B$ is orthogonal.
(d) (3 points) $A$ is not necessarily invertible.
(e) (3 points) The coefficient matrix of the linear transformation

\[
T\begin{pmatrix}x \\ y\end{pmatrix} = \frac{1}{2} \begin{pmatrix} x + y \\ x - y \end{pmatrix}
\]

is orthogonal.
4. Give an example of each of the following, or explain why no such matrix exists.

(a) (5 points) A $4 \times 4$ non-uppertriangular matrix with \( \det(A) = \pi \).

(b) (5 points) A matrix with characteristic polynomial
\[
f_A(\lambda) = (\lambda - 5)^3(\lambda - 2)\lambda.
\]

(c) (5 points) An orthogonal, symmetric, nondiagonal matrix.

(d) (5 points) A nonidentity matrix similar to the identity matrix.

(e) (5 points) A rank 2, nondiagonal, $3 \times 3$ matrix with an eigenbasis.
5. (15 points) Suppose a $5 \times 5$ matrix $A$ satisfies

\[
A \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}.
\]

What can you say about $\det(A)$, $\ker(A)$, $\im(A)$?
6. (a) (10 points) Is
\[ \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{e}_1 = 0, \vec{x} \cdot \vec{e}_2 = 1 \} \]
a subspace of \( \mathbb{R}^n \)?
(b) (10 points) Suppose
\[ V = \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \]
Find matrices \( A \) and \( B \) such that
\[ V = \text{im}(A) \quad \text{and} \quad V = \text{ker}(B). \]
7. (a) (10 points) Compute the area of the 4-gon determined by (1, 1), (−1, 2), (−2, −3), (3, −1).

(b) (10 points) Let

\[ T(\vec{x}) = A\vec{x}, \quad \text{where} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} . \]

Suppose Ω is the unit ball (with volume \( \frac{4}{3}\pi \)). Find the volume of \( T(\Omega) \).
8. (15 points) Suppose $A$ is an $n \times n$ matrix with orthogonal column vectors (but not necessarily orthonormal). Describe the $QR$-factorization of $A$. 
9. (20 points) Find matrices $A$ and $B$ with

$$ f_A(\lambda) = f_B(\lambda) = \lambda^3 - 4\lambda^2 + 4\lambda $$

such that $A$ is diagonalizable and $B$ is not. What are $\det(A)$ and $\det(B)$?
10. (a) (5 points) Find \( a, b, c \in \mathbb{R} \) such that

\[
A = \begin{pmatrix}
5 & -1 & -2 \\
 a & 5 & -2 \\
 b & c & 2
\end{pmatrix}
\]

has an orthonormal eigenbasis.

(b) (10 points) Find an eigenbasis for your solution to (a).

(c) (5 points) Find an orthonormal eigenbasis for your solution to (a).
11. (15 points) Suppose

\[ A = SDS^{-1}, \quad \text{where} \quad D = \begin{pmatrix}
  d_1 & 0 & \cdots & 0 \\
  0 & d_2 & \ddots & \vdots \\
  \vdots & \ddots & \ddots & 0 \\
  0 & \cdots & 0 & d_n
\end{pmatrix}. \]

What can you say about det(A), tr(A), ker(A), im(A)?
12. (15 points) Let

\[ A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \]

Find \( \vec{b} \not\in \text{im}(A) \) such that

\[ \vec{x}^* = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \]

is a least squares solution to the equation

\[ A\vec{x} = \vec{b}. \]