A skein-like multiplication algorithm for unipotent Hecke algebras

Nathaniel Thiem

Abstract

Let $G$ be a finite group of Lie type (e.g. $GL_n(\mathbb{F}_q)$) and $U$ a maximal unipotent subgroup of $G$. If $\psi$ is a linear character of $U$, then the unipotent Hecke algebra is $\mathcal{H}_\psi = \text{End}_{CG}(\text{Ind}_U^G(\psi))$. Unipotent Hecke algebras have a natural basis coming from double cosets of $U$ in $G$. This paper describes relations for reducing products of basis elements, and gives a detailed description of the implications in the case $G = GL_n(\mathbb{F}_q)$.

1 Introduction

Unipotent Hecke algebras interpolate between two classical Hecke algebras, the Gelfand-Graev Hecke algebra [St, Yo1] and the Yokonuma algebra [Yo2] (a generalization of the Iwahori-Hecke algebra). These two classical algebras have not generally been studied from the same perspective, and an underlying philosophy of this paper is that techniques employed in the study of one classical algebra not only apply to the other, but also to all unipotent Hecke algebras.

The Gelfand-Graev Hecke algebra is a commutative algebra that has connections with Chevalley group representation theory [DM], unipotent orbits [Ka1], and Kloosterman sums [CS]. Despite being commutative, computing products in the standard double-coset basis is a challenging problem. The definition of a Hecke algebra implies [CR] that if $T_h$ and $T_k$ are two basis elements, then

$$T_k T_h = \sum_v c_{kh}^v T_v,$$

where

$$c_{kh}^v = \frac{1}{|U|^2} \sum_{u_1, u_2, u_3, u_4 \in U \atop u_1 u_2 = u_3 u_4^{-1}} \psi_{\mu}(u_1^{-1} u_2^{-1} u_3 u_4),$$

but this formula is unhelpful for many applications. Using a geometric approach in [Cu], Curtis analyzed which elements appear in the sums of (*), but computing products in the Gelfand-Graev algebra still remains difficult.

This paper provides a uniform solution to the multiplication problem for Yokonuma Hecke algebras, Gelfand-Graev Hecke algebras, and all unipotent Hecke algebras. The idea is that in a unipotent Hecke algebra the $c_{kh}^v$ in (*) are determined by generalizations of the braid-like relations of the Iwahori-Hecke algebra, and that the multiplication in any unipotent Hecke algebra can be done in a manner directly analogous to the way it is done in the Iwahori-Hecke algebra.

Let $G$ be a finite Chevalley group with a maximal unipotent subgroup $U$. Suppose $\psi_\mu : U \to \mathbb{C}^*$ is a linear character of $U$. Then the unipotent Hecke algebra $\mathcal{H}(G, U, \psi_\mu)$ is

$$\mathcal{H}_\mu = \text{End}_{CG}(\text{Ind}_U^G(\psi_\mu)) \cong e_\mu CG e_\mu,$$

where

$$e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1}) u.$$
Fix a subgroup $N \subseteq G$ of double coset representatives

$$G = \bigcup_{v \in N} UvU, \quad \text{and let} \quad N_\mu = \{ v \in N \mid e_\mu ve_\mu \neq 0 \}.$$

Then the set $\{ e_\mu ve_\mu \mid v \in N_\mu \}$ is a basis for $H_\mu$ [CR, Prop. 11.30].

**Examples.**

1. The Yokonuma Hecke algebra. If $\psi_\mu = 1$ is the trivial character, then $N_1 = N$. Let $W = \langle s_1, s_2, \ldots, s_\ell \rangle$ be the Weyl group of $G$ and $T = \langle h_i(t) \mid 1 \leq i \leq \ell, t \in \mathbb{F}_q^* \rangle$ be a maximal torus so that $N \cong T \times W$. For $w \in W$ and $h \in T$, let $T_{hw} = e_1hwe_1$. By [Yo2], the Yokonuma algebra $H_1$ has a basis $\{ T_v \mid v \in N \}$ with relations

$$T_sT_w = \begin{cases} T_{s,w}, & \text{if } \ell(s,w) = \ell(w) + 1, \\ q^{-1}T_{h(-1)s,w} + q^{-1}\sum_{t \in \mathbb{F}_q^*} T_{h(t)w}, & \text{if } \ell(s,w) = \ell(w) - 1, \quad 1 \leq i \leq \ell, \ w \in W, \\
\end{cases}$$

where if $w = s_1s_2\ldots s_r \in W$ for $r$ minimal, then $\ell(w) = r$. These relations give an “efficient” way to compute arbitrary products $(e_1ue_1)(e_1ve_1)$ in $H_\mu$.

2. The Gelfand-Graev Hecke algebra. If $\psi_\mu$ is in general position, then the Gelfand-Graev module $\text{Ind}_U^G(\psi_\mu)$ is multiplicity free as a $G$-module ([Yo1], [St, Theorem 49]). The corresponding Hecke algebra $H_\mu$ is therefore commutative. However, decomposing the product $(e_\mu ue_\mu)(e_\mu ve_\mu)$ into basis elements is more challenging than in the Yokonuma case [Ch, Cu, Ra].

Section 3 describes some of the subalgebra structure of unipotent Hecke algebras. The main results are in Section 4:

Theorem 4.1 and Corollary 2 give relations similar to those of the Yokonuma algebra (example 1, above) for evaluating the product $(e_\mu ue_\mu)(e_\mu ve_\mu)$, with $u, v \in N_\mu$, in any unipotent Hecke algebra $H_\mu$.

Section 5 applies the main results to the special case when $G = GL_n(\mathbb{F}_q)$, the general linear group over a finite field $\mathbb{F}_q$ with $q$ elements. Readers unfamiliar with the discourse of Chevalley groups may skip ahead to Section 5 (which is independent of Sections 3 and 4).

There are several natural ways to generalize unipotent Hecke algebras. In a series of papers [Ka1, Ka2, Ka3] Kawanaka has analyzed a family of modules obtained by relaxing the maximality condition on $U$. There has also recently been a growing interest in a larger family of characters known as super characters [An, ACDS]. Seeing which aspects of the techniques associated with unipotent Hecke algebras extend to the Hecke algebras of these characters would be an interesting continuation of this work.

**Acknowledgments.** Along with [Th], this paper is part of my Ph.D. thesis. In developing these results, I have enjoyed the supportive environment of the University of Wisconsin-Madison mathematics department, the time supplied by several grants (VIGRE DMS-9819788, NSF DMS-0097977, and NSA MDA904-01-1-0032), and above all the patient help and insights of my advisor Arun Ram.
2 Preliminaries

2.1 Finite Chevalley groups

Let \( g = Z(g) \oplus g_s \) be a reductive Lie algebra, where \( Z(g) \) is the center of \( g \) and \( g_s = [g, g] \) is semisimple. If \( h_s \) is a Cartan subalgebra of \( g_s \), then \( h = Z(g) \oplus h_s \) is a Cartan subalgebra of \( g \). Let

\[
    h^* = \text{Hom}_C(h, \mathbb{C}) \quad \text{and} \quad h_s^* = \text{Hom}_C(h_s, \mathbb{C}).
\]

As an \( h_s \)-module, \( g_s \) decomposes

\[
g_s \cong h_s \oplus \bigoplus_{\alpha \in R} (g_s)_\alpha, \quad \text{where} \quad (g_s)_\alpha = \langle X \in g_s \mid [H, X] = \alpha(H)X, H \in h_s \rangle,
\]

and \( R = \{ \alpha \in h^* \mid \alpha \neq 0, (g_s)_\alpha \neq 0 \} \) is the set of roots of \( g_s \). Choose a set of simple roots \( \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\} \). This choice splits the set of roots \( R \) into positive roots \( R^+ \) and negative roots \( R^- \) with \( R^- = -R^+ \).

For each pair of roots \( \alpha, -\alpha \), there exists a Lie algebra isomorphism \( \phi_\alpha : \mathfrak{s}\mathfrak{l}_2 \to (g_s, g_\alpha) \). Choose these isomorphisms such that if

\[
    X_\alpha = \phi_\alpha \left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \in (g_s)_\alpha, \quad H_\alpha = \phi_\alpha \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \in h_s, \quad X_{-\alpha} = \phi_\alpha \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right) \in (g_s)_{-\alpha},
\]

then \( \{X_\alpha, H_\alpha \mid \alpha \in R, 1 \leq i \leq \ell \} \) is a Chevalley basis of \( g_s \) [Hu, Theorem 25.2].

Let \( V \) be a finite dimensional \( g \)-module such that \( V \) has a \( \mathbb{C} \)-basis \( \{v_1, v_2, \ldots, v_r\} \) that satisfies

(a) There exists a \( \mathbb{C} \)-basis \( \{H_1, \ldots, H_n\} \) of \( h \) such that

1. \( H_\alpha \in \mathbb{Z}_{\geq 0}\text{-span}\{H_1, \ldots, H_n\} \),
2. \( H_i v_j \in \mathbb{Z}v_j \) for all \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, r \).
3. \( \dim_{\mathbb{Z}}(\mathbb{Z}\text{-span}\{H_1, H_2, \ldots, H_n\}) \leq \dim_{\mathbb{C}}(h) \).

(b) \( \frac{X^n}{n!} v_i \in \mathbb{Z}\text{-span}\{v_1, v_2, \ldots, v_r\} \) for \( \alpha \in R, n \in \mathbb{Z}_{\geq 0} \) and \( i = 1, 2, \ldots, r \).

(c) \( \dim_{\mathbb{Z}}(\mathbb{Z}\text{-span}\{v_1, v_2, \ldots, v_r\}) \leq \dim_{\mathbb{C}}(V) \).

(Condition (a) guarantees that \( Z(g) \) acts diagonally. If \( Z(g) = 0 \), then the existence of such a basis is guaranteed by a theorem of Kostant [Hu, Theorem 27.1]).

Let

\[
    h_2 = \mathbb{Z}\text{-span}\{H_1, H_2, \ldots, H_n\}. \tag{2.1}
\]

The finite field \( \mathbb{F}_q \) with \( q \) elements has a multiplicative group \( \mathbb{F}_q^* \) and an additive group \( \mathbb{F}_q^+ \). Let

\[
    V_q = \mathbb{F}_q\text{-span}\{v_1, v_2, \ldots, v_r\}. \tag{2.2}
\]

The finite reductive Chevalley group

\[
    G_V = \langle x_\alpha(a), h_H(b) \mid \alpha \in R, H \in h_2, a \in \mathbb{F}_q, b \in \mathbb{F}_q^* \rangle,
\]

is the subgroup of \( GL(V_q) \) generated by the elements

\[
    x_\alpha(a) = \sum_{n \geq 0} a^n \frac{X^n}{n!}, \quad \text{and} \quad h_H(b) = \text{diag}(b^{\lambda_1(H)}, b^{\lambda_2(H)}, \ldots, b^{\lambda_r(H)}), \quad \text{where} \quad Hv_i = \lambda_i(H)v_i. \tag{2.4}
\]
Remark. If \( g = g_s \), then \( G_V = \langle x_\alpha(t) \mid \alpha \in R, t \in \mathbb{F}_q \rangle \).

Example. Suppose \( g = gl_2 \) and let

\[ V = \mathbb{C}\text{-span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \]

be the natural \( g \)-module given by matrix multiplication. Then \( h \) has a basis

\[ h = \left\{ \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \mid a, b \in \mathbb{C} \right\} = \mathbb{C}\text{-span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \]

By direct computation,

\[ x_\alpha(t) = \begin{pmatrix} 1 \\ t \\ 0 \end{pmatrix} \quad \text{and} \quad h_{\begin{pmatrix} a \\ 0 \\ b \end{pmatrix}}(t) = \begin{pmatrix} t^a \\ 0 \\ t^b \end{pmatrix} \quad \text{for } a, b \in \mathbb{Z}, \]

and \( G_V = GL_2(\mathbb{F}_q) \) (the general linear group).

### 2.2 Important subgroups of a Chevalley group

Let \( G = G_V \) be a Chevalley group defined with a \( g \)-module \( V \) as above. The group \( G \) contains a subgroup \( U \) given by

\[ U = \langle x_\alpha(t) \mid \alpha \in R^+, t \in \mathbb{F}_q \rangle, \]

which decomposes as

\[ U = \prod_{\alpha \in R^+} U_\alpha, \quad \text{where} \quad U_\alpha = \langle x_\alpha(t) \mid t \in \mathbb{F}_q \rangle, \]

with uniqueness of expression for any fixed ordering of the positive roots [St, Lemma 18]. For each \( \alpha \in R^+ \), the map

\[ U_\alpha \xrightarrow{\sim} \mathbb{F}_q^+, \quad x_\alpha(t) \mapsto t \]

is a group isomorphism.

For \( \alpha, \beta \in R \), define the maps

\[ s_\alpha : h^* \xrightarrow{\sim} h^* \quad \text{and} \quad s_\alpha : h = Z(g) \oplus h_s \xrightarrow{\sim} h \]

with \( H = H_\beta \) and \( H = H_\beta - \beta(H_\alpha)H_\alpha \). \hspace{1cm} (2.5)

The Weyl group of \( G \) is \( W = \langle s_\alpha \mid \alpha \in R \rangle \) and has a presentation

\[ W = \langle s_1, s_2, \ldots, s_\ell \mid s_i^2 = 1, (s_is_j)^{m_{ij}} = 1, 1 \leq i \neq j \leq \ell, \quad m_{ij} \in \mathbb{Z}_{>0}, \quad s_i = s_{\alpha_i}. \]

If \( w = s_{i_1}s_{i_2}\cdots s_{i_r} \) with \( r \) minimal, then the length \( \ell(w) = r \).

Let \( h_\mathbb{Z} \) be as in (2.1). If \( q > 3 \), then the subgroup

\[ T = \langle h_H(t) \mid H \in h_\mathbb{Z}, t \in \mathbb{F}_q^+ \rangle \]

has its normalizer in \( G \) given by

\[ N = \langle w_\alpha(t), h \mid \alpha \in R, h \in T, t \in \mathbb{F}_q^+ \rangle, \quad \text{where} \quad w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t). \]

If \( \alpha \in R \), then \( h_{H_\alpha}(t) = w_\alpha(t)w_\alpha(1)^{-1} \). Write \( h_\alpha(t) = h_{H_\alpha}(t) \) and \( h_i(t) = h_{\alpha_i}(t) \).
There is a natural surjection from $N$ onto the Weyl group $W$ with kernel $T$ given by

\[
\begin{align*}
\pi : & \quad N \rightarrow W \\
& w_\alpha(t) \mapsto s_\alpha, \quad \text{for } \alpha \in R, t \in F_q^*, \\
& h \mapsto 1, \quad \text{for } h \in T.
\end{align*}
\]

(2.6)

Suppose $v \in N$. Then for each minimal expression

\[
\pi(v) = s_{i_1}s_{i_2}\ldots s_{i_r}, \quad \text{with } \ell(\pi(v)) = r,
\]

there is a unique decomposition of $v$ as

\[
v = v_1v_2\ldots v_r, \quad \text{where } v_k = w_i(1) \text{ and } v_r \in T.
\]

(2.7)

Write

\[
\xi_i = w_i(1).
\]

(2.8)

### 2.3 Unipotent Hecke algebras

Let $G$ be a finite Chevalley group. Fix a nontrivial homomorphism $\psi : F_q^+ \rightarrow \mathbb{C}^*$. If

\[
\begin{align*}
\mu : & \quad \alpha \rightarrow \mu_\alpha \\
& \alpha \in R_+ \rightarrow \mathbb{F}_q^* \\
& \text{satisfies } \mu_\alpha = 0 \text{ for all } \alpha \text{ not simple,}
\end{align*}
\]

(2.9)

then the map

\[
\psi_\mu : U \rightarrow \mathbb{C}^*
\]

\[
x_\alpha(t) \mapsto \psi(\mu_\alpha t)
\]

(2.10)

is a linear character of $U$. With the exception of a few degenerate special cases of $G$ (which can be avoided if $q > 3$), all linear characters of $U$ are of this form [Yo1.5, Theorem 1].

The unipotent Hecke algebra $\mathcal{H}(G,U,\psi_\mu)$ is

\[
\mathcal{H}_\mu = \text{End}_{\mathbb{C}G}(\text{Ind}_U^G(\psi_\mu)),
\]

(2.11)

or viewed as a subset of $\mathbb{C}G$,

\[
\mathcal{H}_\mu = e_\mu \mathbb{C}Ge_\mu, \quad \text{where } e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1})u.
\]

(2.12)

**Remark:** Since $T$ is in the normalizer of $U$ in $G$, $T$ acts on the linear characters of $U$ by

\[
h \chi(u) = \chi(huh^{-1}), \quad \text{where } u \in U, \; h \in T, \; \text{and } \chi : U \rightarrow \mathbb{C}^*.
\]

If two linear characters $\chi$ and $\gamma$ are in the same $T$-orbit then $\mathcal{H}(G,U,\chi) \cong \mathcal{H}(G,U,\gamma)$ (the converse does not necessarily hold).

The group $G$ has a double-coset decomposition

\[
G = \bigsqcup_{v \in N} UvU, \quad \text{[St, Theorem 4 and } B = UT]\]

(2.13)

and if

\[
N_\mu = \{v \in N \mid e_\mu ve_\mu \neq 0\}
\]

\[
= \{v \in N \mid u, vuv^{-1} \in U \text{ implies } \psi_\mu(u) = \psi_\mu(vuv^{-1})\}
\]

(2.14)

then the set \{\text{e}_\mu ve_\mu \mid v \in N_\mu\} is a basis for $\mathcal{H}_\mu$ [CR, Prop. 11.30].

**Examples** (see also the Introduction).
1. The Yokonuma Hecke algebra. If $\mu_\alpha = 0$ for all positive roots $\alpha$, then $\psi_\mu = 1$ is the trivial character and $N_1 = N$. Let $T_v = e_v v e_1$ for $v \in N$, with $T_i = T_{\xi_i}$ (as in (2.8)) and $T_H(t) = T_{\xi_H}(t)$. If $v \in N$ has a decomposition $v = v_1 v_2 \cdots v_r v_T$ (as in (2.7)), then

$$T_v = T_{i_1} T_{i_2} \cdots T_{i_r} T_{v_T}.$$ 

Thus, the Yokonuma algebra $\mathcal{H}_1$ has generators $\{T_i, T_h \mid 1 \leq i \leq \ell, h \in T\}$ (see [Yo2]) with relations,

$$T_i^2 = q^{-1} T_{H_{\alpha_i}}(-1) + q^{-1} \sum_{t \in F_q} T_{H_{\alpha_i}}(t^{-1}) T_i, \quad 1 \leq i \leq \ell,$$

$$\overset{m_{ij}}{\underset{m_{ij}}{\underbrace{T_i T_j T_i \cdots = T_j T_i T_j \cdots}},} \quad (s_i s_j)^m = 1,$$

$$T_i T_h = T_{s_i h} T_i, \quad h \in T,$$

$$T_h T_k = T_{h k}, \quad h, k \in T.$$

These relations give an “efficient” way to compute arbitrary products $T_u T_v$ in $\mathcal{H}_1$. There is a surjective map from the Yokonuma algebra onto the Iwahori-Hecke algebra that sends $T_h \mapsto 1$ for all $h \in T$. “Setting $T_h = 1$” in the Yokonuma algebra relations recovers relations for the Iwahori-Hecke algebra,

$$T_i^2 = q^{-1} + q^{-1} (q - 1) T_i,$$

$$\overset{m_{ij}}{\underbrace{T_i T_j \cdots = T_j T_i \cdots}}.$$ 

Furthermore, there is a surjective map from the Iwahori Hecke algebra onto the group algebra of the Weyl group given by $T_i \mapsto s_i$ and $q \mapsto 1$. Thus, by “setting $T_i = s_i$ and $q = 1$” we retrieve the Coxeter relations of $W$,

$$s_i^2 = 1, \quad s_i s_j s_i \cdots = s_j s_i s_j \cdots.$$ 

2. The Gelfand-Graev Hecke algebra. By definition, if $\mu_\alpha \neq 0$ for all simple roots $\alpha$, then $\psi_\mu$ is in general position. The Gelfand-Graev Hecke algebra $\mathcal{H}_\mu$ is commutative ([Yo1],[St, Theorem 49]).

3. Parabolic subalgebras of $\mathcal{H}_\mu$

Let $\psi : U \to G$ be as in (2.10). Fix a subset $J \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ such that

$$\text{if } \mu_\alpha \neq 0, \text{ then } \alpha_i \in J. \quad (3.1)$$

For example, if $\psi_\mu$ is in general position, then $J = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$, but if $\psi_\mu$ is trivial, then $J$ could be any subset.

Let

$$W_J = \langle s_i \in W \mid \alpha_i \in J \rangle, \quad P_J = \langle U, T, W_J \rangle \quad \text{and} \quad R_J = \mathbb{Z}-\text{span}\{\alpha_i \in J\} \cap R.$$ 

Then $P_J$ has subgroups

$$L_J = \langle T, W_J, U_\alpha \mid \alpha \in R_J \rangle \quad \text{and} \quad U_J = \langle U_\alpha \mid \alpha \in R^+ - R_J \rangle \quad (3.2)$$
(a Levi subgroup and the unipotent radical of \( P_J \), respectively). Note that

\[ U_J L_J = P_J, \quad U_J \cap L_J = 1, \quad \text{and, in fact,} \quad P_J = U_J \ltimes L_J. \]

Define the idempotents of \( CU \),

\[ e_{\mu J} = \frac{1}{|L_J \cap U|} \sum_{u \in L_J \cap U} \psi_{\mu} (u^{-1})u \quad \text{and} \quad e'_{J} = \frac{1}{|U_J|} \sum_{u \in U_J} u, \quad \tag{3.3} \]

so that \( e_{\mu} = e_{\mu J} e'_{J} \).

The group homomorphisms

\[ P_J \rightarrow L_J \quad \text{and} \quad P_J \rightarrow G \quad \text{for} \quad l \in L_J, \ u \in U_J, \]

induce functors

\[ \text{Ind}^P_J : \{ L_J \text{-modules} \} \rightarrow \{ P_J \text{-modules} \} \quad \text{and} \quad \text{Ind}^G_P : \{ P_J \text{-modules} \} \rightarrow \{ G \text{-modules} \} \quad \text{M} \rightarrow e'_{J} M \quad \text{and} \quad \text{M}' \rightarrow CG \otimes_{CP_J} M' \]

whose composition is the functor \( \text{Ind}^G_{L_J} \). In the special case when \( (CL_J)e \) is an \( L_J \)-module with corresponding idempotent \( e \),

\[ \text{Ind}^G_{L_J} : \{ L_J \text{-modules} \} \rightarrow \{ G \text{-modules} \} \quad CL_J e \rightarrow CG e'_{J}. \]

The map \( \psi_{\mu} : U \rightarrow C^* \) restricts to a linear character \( \text{Res}^{U}^{L_J} \psi_{\mu} : L_J \cap U \rightarrow C^* \). To make the notation less heavy-handed, write \( \psi_{\mu} : L_J \cap U \rightarrow C^* \), for \( \text{Res}^{U}^{L_J} \psi_{\mu} \).

**Lemma 3.1.** Let \( \psi_{\mu} \) be as in (2.10). Then

\[ \text{Ind}^G_U (\psi_{\mu}) \cong \text{Ind}^G_{L_J} (\text{Ind}^{L_J}_{U \cap L_J} (\psi_{\mu})). \]

**Proof.** Recall \( \text{Ind}^G_U (\psi_{\mu}) \cong CG e_{\mu} \). On the other hand,

\[ \text{Ind}^{L_J}_{U \cap L_J} (\psi_{\mu}) \cong CL_J e_{\mu J} \quad \text{implies} \quad \text{Ind}^G_{L_J} (\text{Ind}^{L_J}_{U \cap L_J} (\psi_{\mu})) \cong CG e_{\mu J} e'_{J}, \]

where \( e_{\mu J} \) is as in (3.3). But \( e_{\mu J} e'_{J} = e_{\mu} \), so

\[ \text{Ind}^G_U (\psi_{\mu}) \cong CG e_{\mu} \cong CG e_{\mu J} e'_{J} \cong \text{Ind}^G_{L_J} (\text{Ind}^{L_J}_{U \cap L_J} (\psi_{\mu})). \]

\[ \square \]

**Theorem 3.1.** The map

\[ \theta : \text{End}_{CL_J} (\text{Ind}^{L_J}_{U \cap L_J} (\psi_{\mu})) \rightarrow \mathcal{H}_{\mu}, \quad e_{\mu J} v e_{\mu J} \mapsto e_{\mu} v e_{\mu}, \quad \text{for} \quad v \in L_J \cap N_{\mu}, \]

is an injective algebra homomorphism.

**Proof.** Let \( v \in N \cap L_J \). Since \( e_{\mu J} v e_{\mu J} \in CL_J, \ e'_{J} \in CU_J \) and \( L_J \cap U_J = 1 \),

\[ e_{\mu J} v e_{\mu J} = 0 \quad \text{if and only if} \quad e'_{J} e_{\mu J} v e_{\mu J} = 0. \]
Because $L_J$ normalizes $U_J$, both $e_{\mu J}$ and $v$ commute with $e'_J$. Therefore,
\[ e_{\mu J}ve_{\mu J} = 0 \quad \text{if and only if} \quad e'_Je_{\mu J}ve'_J = e_{\mu J}ve_{\mu J} = 0, \]
and
\[ \{e_{\mu J}ve_{\mu J} \neq 0\} \quad \text{if and only if} \quad v \in N_{\mu} \cap L_J. \]
Consequently, $\theta$ is both well-defined and injective.

Consider $\theta(e_{\mu J}ue_{\mu J})$.
\[ \theta(e_{\mu J}ue_{\mu J}) = e_{\mu J}ue_{\mu J} = e_{\mu J}ue_{\mu J}e'_{\mu J}ve'_{\mu J}. \]
Since $u$ commutes with $e'_{\mu J}$,
\[ \theta(e_{\mu J}ue_{\mu J}) = e_{\mu J}e'_{\mu J}ue_{\mu J}ve'_{\mu J}e_{\mu J} = \theta(e_{\mu J}ue_{\mu J}ve_{\mu J}). \]

Write
\[ L_J = \theta(End_{C_{L_J}}(Ind_{U \cap L_J}(\psi_{\mu}))) \subseteq H_{\mu} \quad (3.4) \]
The $L_J$ are “parabolic” subalgebras of $H_{\mu}$, in that they have a similar role in the representation theory of $H_{\mu}$ as parabolic subgroups $P_J$ have in the representation theory of $G$.

### 3.1 Weight space decompositions for $H_{\mu}$-modules

An important special case of Theorem 3.1 is when
\[ J = J_{\mu} = \{\alpha_i \text{ simple root} \mid \mu_{\alpha_i} \neq 0\}, \]
so that $J_{\mu}$ is maximal satisfying (3.1). Write $L_{\mu} = L_{J_{\mu}}$, $W_{\mu} = W_{J_{\mu}}$, etc.

**Corollary 1.** The algebra $L_{\mu}$ is a nonzero commutative subalgebra of $H_{\mu}$.

**Proof.** As a character of $U \cap L_{\mu}$, $\psi_{\mu}$ is in general position, so $Ind_{U \cap L_{\mu}}^{L_{\mu}}(\psi_{\mu})$ is a Gelfand-Graev module and $L_{\mu}$ is a Gelfand-Graev Hecke algebra (see example 2 in Section 2.3). Since $L_{\mu}$ is commutative, all the irreducible $L_{\mu}$-modules are one-dimensional. Let $\hat{L}_{\mu}$ be an indexing set for the irreducible modules of $L_{\mu}$. Suppose $V$ is an $H_{\mu}$-module. Since $L_{\mu} \cong End_{C_{L_{\mu}}}(Ind_{U \cap L_{\mu}}^{L_{\mu}}(\psi_{\mu}))$, $L_{\mu}$ is semisimple, and as an $L_{\mu}$-module,
\[ V \cong \bigoplus_{\gamma \in \hat{L}_{\mu}} V_{\gamma} \quad \text{where} \quad V_{\gamma} = \{v \in V \mid xv = \gamma(x)v, x \in L_{\mu}\}. \]

If $\gamma \in \hat{L}_{\mu}$, then $V_{\gamma}$ is the $\gamma$-weight space of $V$, and $\gamma$ is a weight of $V$ if $V_{\gamma} \neq 0$.

**Examples.**

1. In the Yokonuma algebra $\psi_{\mu} = 1$, $J_{\mu} = \emptyset$ and $L_{1} = e_{1}CTe_{1} \cong CT$.

2. In the Gelfand-Graev Hecke algebra case, $J_{\mu} = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ and $L_{\mu} = H_{\mu}$.

**Remark.** Since dim($V_{\gamma}$) can be greater than one, $L_{\mu}$ is not in general a maximal commutative subalgebra of $H_{\mu}$.
4 Multiplication of basis elements

This section examines the decomposition of products in terms of the natural basis

\[(e_\mu v^e_\mu)(e_\mu v^e_\mu) = \sum_{\nu' \in N_\mu} c'_{uv} (e_\mu v^e_\mu).\]

In particular, Theorem 4.1, below, gives a set of braid-like relations (similar to those of the Yokonuma algebra) for manipulating the products, and Corollary 2 gives a recursive formula for computing these products.

4.1 Chevalley group relations

The relations governing the interaction between the subgroups \(N, U,\) and \(T\) will be critical in describing the Hecke algebra multiplication in the following section. They can all be found in [St, §3].

The subgroup

\[U = \langle x_\alpha(t) \mid \alpha \in R^+, t \in \mathbb{F}_q \rangle\]

has generators \(\{x_\alpha(t) \mid \alpha \in R^+, t \in \mathbb{F}_q\},\) with relations

\[x_\alpha(a)x_\beta(b)x_\alpha(a)^{-1}x_\beta(b)^{-1} = \prod_{\gamma = \alpha + j\beta \in R^+, i,j \in \mathbb{Z}_{\geq 0}} x_\gamma(z_i \alpha^i \beta^j), \quad (U1)\]

\[x_\alpha(a)x_\alpha(b) = x_\alpha(a + b), \quad (U2)\]

where \(z_\gamma \in \mathbb{Z}\) depends on \(i, j, \alpha, \beta,\) but not on \(a, b \in \mathbb{F}_q\) [St, Lemma 15]. The \(x_\gamma\) have been explicitly computed for various types in [De, St].

The subgroup \(N\) has generators \(\{\xi_i, h_H(t) \mid i = 1, 2, \ldots, \ell, H \in \mathfrak{h}_\mathbb{Z}, t \in \mathbb{F}_q^*\},\) with relations

\[\xi_i^2 = h_i(-1), \quad (N1)\]

\[\xi_i^{s_i} = \xi_j^{s_j} \cdots = \xi_i^{s_i} = \xi_j^{s_j} \cdots, \quad \text{where } (s_is_j)^{m_{ij}} = 1 \text{ in } W, \quad (N2)\]

\[\xi_i h_H(t) = h_{s_i(H)}(t)\xi_i, \quad (N3)\]

\[h_H(a)h_H(b) = h_H(ab), \quad (N4)\]

\[h_H(a)h_H(b) = h_H(b)h_H(a), \quad \text{for } H, H' \in \mathfrak{h}, \quad (N5)\]

\[h_H(a)h_{H'}(a) = h_{H+H'}(a), \quad \text{for } H, H' \in \mathfrak{h}, \quad (N6)\]

\[h_{H_1}(t_1)h_{H_2}(t_2) \cdots h_{H_k}(t_k) = 1, \quad \text{if } t_1^{\lambda_1(H_1)} \cdots t_k^{\lambda_k(H_k)} = 1 \text{ for all } 1 \leq j \leq r, \quad (N7)\]

where \(\lambda_j : \mathfrak{h} \to \mathbb{C}\) depends on \(V\) as in (2.4).

The double-coset decomposition of \(G\) (2.13) implies \(G = \langle U, N \rangle.\) Thus, \(G\) is generated by \(\{x_\alpha(a), \xi_i, h_H(b) \mid \alpha \in R^+, a \in \mathbb{F}_q, i = 1, 2, \ldots, \ell, H \in \mathfrak{h}_\mathbb{Z}, b \in \mathbb{F}_q^*\}\) with relations \((U1)-(N7)\) and

\[\xi_i x_\alpha(t) \xi_i^{-1} = x_{s_i(\alpha)}(c_{i\alpha}t), \quad \text{where } c_{i\alpha} = \pm 1 \quad (\text{Un1})\]

\[h x_\alpha(b) h^{-1} = x_\alpha(\alpha(h)b), \quad \text{for } h \in T, \quad (\text{Un2})\]

\[\xi_i x_i(t) \xi_i = x_i(t^{-1})h_i(t^{-1})\xi_i x_i(t^{-1}), \quad \text{where } x_i(t) = x_{\alpha_i}(t) \text{ and } t \neq 0, \quad (\text{Un3})\]

where for \(\alpha \in R\) and \(h_H(t) \in T,\)

\[\alpha(h_H(t)) = t^{\alpha(H)}. \quad (4.1)\]
Fix a $\psi_\mu : U \to \mathbb{C}^*$ as in (2.10). For $k \in \mathbb{F}_q$, let
\[
e_\alpha(k) = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\mu_\alpha kt)x_\alpha(t) \quad \text{with the convention} \quad e_\alpha = e_\alpha(1).
\]
(E2) Suppose $\alpha \neq \alpha$. In particular, given any $\alpha \in R^+$, we may choose the ordering of the positive roots to have $e_\alpha$ appear either first or last. Therefore, since $e_\alpha$ is an idempotent,
\[
e_\mu e_\alpha = e_\mu = e_\alpha e_\mu.
\]
(E4) Using relation (UN1),
\[
R_w = \{ \alpha \in R^+ \mid w(\alpha) \in R^- \} = \{ \alpha_{i_1}, s_{i_2}(\alpha_{i_2-1}), \ldots, s_{i_r}, s_{i_{r-1}} \cdots s_{i_2}(\alpha_i) \},
\]
where the second equality is from [Bo, VI.1, Corollary 2 of Proposition 17].

Lemma 4.1. Let $v \in N$, and let $w = \pi(v)$ (with $\pi : N \to W$ as in (2.6)).
\[
\begin{align*}
\xi_i e_\alpha(k) \xi_i^{-1} &= e_{s_i \alpha}(c_{i\alpha}k), & \text{for } \alpha \in R^+, 1 \leq i \leq n - 1, \quad \text{(E1)} \\
v e_\alpha v^{-1} &= e_{wa}, & \text{for } \alpha \notin R_w, v \in N_\mu, \quad \text{(E2)} \\
h e_\alpha(k) h^{-1} &= e_\alpha(ka(h)^{-1}), & \text{for } h \in T, w = \pi(v), \quad \text{(E3)} \\
eg_\mu x_\alpha(t) &= \psi(\mu_\alpha t)e_\mu = x_\alpha(t) e_\mu. \quad \text{(E4)}
\end{align*}
\]
Proof. (E1) Using relation (UN1),
\[
\xi_i e_\alpha(k) \xi_i^{-1} = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\mu_\alpha kt)x_\alpha(t) \xi_i^{-1} = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\mu_\alpha kt)x_{s_i \alpha}(c_{i\alpha}t)
\]
\[
= \frac{1}{q} \sum_{t' \in \mathbb{F}_q} \psi(-\mu_\alpha c_{i\alpha}t')x_{s_i \alpha}(t') = e_{s_i \alpha}(c_{i\alpha}k).
\]
(E2) Suppose $\alpha \notin R_w$. Since $v \in N_\mu$,
\[
\psi(\mu_\alpha t) = \psi_\mu(x_\alpha(t)) = \psi_\mu(v x_\alpha(t)v^{-1}) = \psi_\mu(x_{wa}(kt))
\]
\[
= \psi(\mu_{wa} kt), \quad \text{for some } k \in \mathbb{Z}_{\neq 0}.
\]
In particular, since $\psi$ is nontrivial, $\mu_\alpha = k \mu_{wa}$. Thus,
\[
v e_\alpha v^{-1} = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\mu_\alpha t)x_{wa}(kt) = \frac{1}{q} \sum_{t' \in \mathbb{F}_q} \psi(-\mu_\alpha k^{-1}t')x_{wa}(t') = e_{wa}.
\]
(E3) Since $hx_\alpha(t)h^{-1} = x_\alpha(\alpha(h)t)$,
\[
h e_\alpha(k) h^{-1} = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-kt)x_\alpha(\alpha(h)t) = \sum_{t \in \mathbb{F}_q} \psi(-kt\alpha(h)^{-1})x_\alpha(t) = e_\alpha(ka(h)^{-1}).
\]
Note that
\[ e_\alpha x_\alpha(t) = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \psi(-\mu_\alpha a)x_\alpha(a + t) = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \psi(-\mu_\alpha a)x_\alpha(a + t) \]
\[ = \frac{1}{q} \sum_{a' \in \mathbb{F}_q} \psi(-\mu_\alpha (a' - t))x_\alpha(a') = \frac{1}{q} \sum_{a' \in \mathbb{F}_q} \psi(-\mu_\alpha a')\psi(\mu_\alpha t)x_\alpha(a') \]
\[ = \psi(\mu_\alpha t)e_\alpha \]

Therefore, by (4.4), \( e_\mu x_\alpha(t) = e_\mu e_\alpha x_\alpha(t) = e_\mu x_\alpha(t) \).

\[ \square \]

### 4.2 Local Hecke algebra relations

Let \( u = u_1u_2 \cdots u_r u_T \in N \) decompose according to \( s_1, s_2, \ldots, s_r \in W \). For \( 1 \leq k \leq r \) define constants \( c_k = \pm 1 \) and roots \( \beta_k \in R^+ \) by the equation
\[
x_{\beta_k}(c_k t) = (u_{k+1} \cdots u_r)^{-1}x_{\alpha_k}(t)(u_{k+1} \cdots u_r). \tag{4.6}
\]

Note that \( R_{\pi(u)} = \{ \beta_1, \beta_2, \ldots, \beta_r \} \) (see (4.5)). Define \( f_u \in \mathbb{F}_q[y_1, y_2, \ldots, y_r] \) by
\[
f_u = -\frac{\mu_{\beta_1}c_1}{\beta_1(u_T)} y_1 - \frac{\mu_{\beta_2}c_2}{\beta_2(u_T)} y_2 - \cdots - \frac{\mu_{\beta_r}c_r}{\beta_r(u_T)} y_r, \tag{4.7}
\]

and for \( k = 1, 2, \ldots, r \), let
\[
u_k(t) = \xi_{i_k}(t), \quad \text{where} \quad \xi_i(t) = \xi_i x_i(t). \tag{4.8}
\]

**Theorem 4.1.** Let \( u = u_1u_2 \cdots u_r u_T, v = v_1v_2 \cdots v_s v_T \in N_\mu \) decompose according to \( s_1, s_2, \ldots, s_r \in W \) and \( s_1, s_2, \ldots, s_s \in W \), respectively, as in (2.7). Then

(a)
\[
(e_\mu u e_\mu)(e_\mu v e_\mu) = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f_u)(t) e_\mu(u_1(t)u_2(t) \cdots u_r(t)) (v_1v_2 \cdots v_s) h e_\mu,
\]

where \( h = v_T v^{-1} u_T v \in T \).

(b) The following local relations suffice to compute the product \((e_\mu u e_\mu)(e_\mu v e_\mu)\).

\[
\sum_{t \in \mathbb{F}_q^r} (\psi \circ f)(t) \xi_i(t) \xi_i = (\psi \circ f)(0) \xi_i^2 + \sum_{t \in \mathbb{F}_q} (\psi \circ f)(t) x_\alpha(t)^{-1} \xi_i h_\alpha(t) x_\alpha(t), \tag{H1}
\]
\[
\xi_i x_\alpha(t) = x_{s_i(\alpha)}(c_\alpha t) \xi_i, \tag{H2}
\]
\[
x_\alpha(t) h = h x_\alpha(\alpha h^{-1} t), \tag{H3}
\]
\[
e_\mu x_\alpha(t) = \psi(\mu_\alpha t)e_\mu = x_\alpha(t)e_\mu, \tag{H4}
\]
\[
(\psi \circ f)(t)(\psi \circ g)(t) = (\psi \circ (f \circ g))(t), \quad \text{for} \quad f, g \in F_q[y_1^\pm, \ldots, y_r^\pm], \quad t \in \mathbb{F}_q^r, \tag{H5}
\]
\[
h_\alpha(t) \xi_i = \xi_i h_{s_i(\alpha)}(t), \tag{H6}
\]
\[
\xi_i(a) x_\alpha(b) = \prod_{\gamma = \alpha} x_{s_\gamma}^{-1}(c_{s_\gamma}(\gamma) z_{s_\gamma}^{-1} b^m) \xi_i(a), \quad \text{where} \quad \alpha \neq \alpha_i, \tag{H7}
\]
\[
\sum_{a \in \mathbb{F}_q} \Phi(a) \xi_i(a) x_i(b) = \sum_{a \in \mathbb{F}_q} \Phi(a - b) \xi_i(a), \quad \text{for some map} \quad \Phi : F_q \rightarrow CG, \tag{H8}
\]
\[
h_\alpha(a) h_\alpha(b) = h_\alpha(ab), \tag{H9}
\]
\[
\underbrace{\xi_i \xi_j \xi_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{\xi_i \xi_j \xi_i \cdots}_{m_{ij} \text{ terms}}, \quad \text{where} \quad m_{ij} \text{ is the order of} \ s_i s_j \text{ in} \ W \tag{H10}
\]
Proof. (a) Order the positive roots so that by (4.3)

\[ e_\mu u e_\mu v e_\mu = e_\mu u \left( \prod_{\alpha \notin R_w} e_\alpha \right) e_{\beta_1} e_{\beta_2} \cdots e_{\beta_r} v e_\mu \]  
(definition $\beta_k$)

\[ = e_\mu u \left( \prod_{\alpha \notin R_w} e_\alpha \right) u e_{\beta_1} e_{\beta_2} \cdots e_{\beta_r} v e_\mu \]  
(Lemma 4.1,E2)

\[ = e_\mu u e_{\beta_1} e_{\beta_2} \cdots e_{\beta_r} v e_\mu \]  
(Lemma 4.1, E4)

\[ = e_\mu u_1 u_2 \cdots u_r u_T e_{\beta_1} e_{\beta_2} \cdots e_{\beta_r} v e_\mu \]  
(Lemma 4.1,E3)

\[ = e_\mu u_1 u_2 \cdots u_r e_{\beta_1} \left( \frac{\mu_{\beta_1}}{\tau_1(u_T)} \right) e_{\beta_2} \left( \frac{\mu_{\beta_2}}{\tau_2(u_T)} \right) \cdots e_{\beta_r} \left( \frac{\mu_{\beta_r}}{\tau_r(u_T)} \right) u_T v e_\mu \]  
(Lemma 4.1,E1)

\[ = e_\mu u_1 e_{\alpha_1} \left( \frac{\mu_{\beta_1}}{\tau_1(u_T)} \right) u_2 e_{\alpha_2} \left( \frac{\mu_{\beta_2}}{\tau_2(u_T)} \right) \cdots u_r e_{\alpha_r} \left( \frac{\mu_{\beta_r}}{\tau_r(u_T)} \right) v_1 \cdots v_s v_T v^{-1} u_T v e_\mu \]  
(definition $e_\alpha$)

\[ = \frac{1}{q^r} \sum_{t_1, \ldots, t_r \in \mathbb{F}_q} \psi(\cdots) e_\mu u_1(t_1) \cdots u_r(t_r) v_1 \cdots v_s h e_\mu, \]  
(by (H5))

where $h = v_T v^{-1} u_T v \in T$, as desired.

(b) First, note that these relations are in fact correct (though not necessarily sufficient): (H1) comes from (UN3); (H2) comes from (UN1); (H3) comes from (UN2); (H4) comes from (E4); (H5) comes from the multiplicativity of \( \psi \); (H6) comes from (N3); (H7) comes from (U1) and (UN1); (H8) comes from (U2); (H9) is (N4); and (H10) is (N2). It therefore remains to show sufficiency.

By (a) we may write

\[ (e_\mu u e_\mu)(e_\mu v e_\mu) = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q} (\psi \circ f)(t)e_\mu u_1(t_1) \cdots u_r(t_r) v_1 \cdots v_s h e_\mu \]

for some $f \in \mathbb{F}_q[y_1, \ldots, y_r]$ and $h \in T$. Say $t_k$ is resolved if the only part of the sum depending on $t_k$ is $\psi \circ f$. The product is reduced when all the $t_k$ are resolved. I will show how to resolve $t_r$ and the result will follow by induction.

Use relation (H2) to define the constant $d$ and the root $\gamma \in R$ by

\[ (v_1 v_2 \cdots v_s)^{-1} x_{\alpha_{\ell}}(t)(v_1 v_2 \cdots v_s) = x_\gamma(dt) \]  
(4.9)

where $\ell(\pi(v)) = s$.

There are two possible situations:

Case 1. $\gamma \in R^+$,

Case 2. $\gamma \in R^-$.
In Case 1,

\[(e_\mu v_\mu)(e_\mu v_\mu) = \frac{1}{q^r} \sum_{t \in \mathbb{F}^r_q} \left( \psi \circ f \right)(t)e_\mu u_1(t_1) \cdots u_r x_i,_{r}(t_r)v_1 \cdots v_s h e_\mu \]  
(by (a))

\[= \frac{1}{q^r} \sum_{t \in \mathbb{F}^r_q} \left( \psi \circ f \right)(t)e_\mu u_1(t_1) \cdots u_r(t_{r-1})u_r v_1 \cdots v_s x_r(d t_r) h e_\mu \]  
(by (H2))

\[= \frac{1}{q^r} \sum_{t \in \mathbb{F}^r_q} \left( \psi \circ f \right)(t)e_\mu u_1(t_1) \cdots u_r(t_{r-1})u_r v_1 \cdots v_s h x_r(d \gamma(h)^{-1} r) e_\mu \]  
(by (H3))

\[= \frac{1}{q^r} \sum_{t \in \mathbb{F}^r_q} \left( \psi \circ f \right)(t)e_\mu u_1(t_1) \cdots u_r(t_{r-1})u_r v_1 \cdots v_s h \psi(\mu \gamma(h)^{-1} r) e_\mu \]  
(by (H4))

\[= \frac{1}{q^r} \sum_{t \in \mathbb{F}^r_q} \left( \psi \circ f \right)(t)e_\mu u_1(t_1) \cdots u_r(t_{r-1})u_r v_1 \cdots v_s h \psi(\mu \gamma(h)^{-1} y_r)(t_r) e_\mu \]  
(by (H5))

where \(g = f + \mu \gamma d \gamma(h)^{-1} y_r\). We have resolved \(t_r\) in Case 1.

In Case 2, \(\gamma \in R^+\), so we can no longer move \(x_i,_{r}(t_r)\) past the \(v_j\). Instead,

\[(e_\mu v_\mu)(e_\mu v_\mu) \]  

\[= \frac{e_\mu}{q^r} \sum_{t \in \mathbb{F}^r_q} \left( \psi \circ f \right)(t_1) \cdots u_r(t_{r-1})u_r v_1 \cdots v_s h e_\mu \]  
(by (H1))

\[+ \frac{e_\mu}{q^r} \sum_{t' \in \mathbb{F}^r_q} \left( \psi \circ f \right)(t',0) u_1(t_1) \cdots u_r(t_{r-1}) \sum_{t_r \in \mathbb{F}^r_q} \left( \psi \circ f \right)(t_r, t_r x_i,_{r}(t_r)^{-1}) u_r v_1 \cdots v_s h e_\mu \]  
(by (H2,H3,H4))

where \(g = f - \mu \gamma d \gamma(h)^{-1} y_r\) (same as in the analogous steps in Case 1).
Lemma 4.2 (Resolving $t_k$). Let $u = u_1u_2 \cdots u_k \in N$ decompose according to $s_{i_1}s_{i_2} \cdots s_{i_k}$ (with $u_T = 1$). Suppose $v \in N$ and $f \in \mathbb{F}_q[y_1, y_2, \ldots, y_k]$. Define $\gamma \in R$ and $d \in \mathbb{C}$ by the equation $v^{-1}x_{i_k}(t)v = x_\gamma(dt)$. Then

**Case 1** If $\ell(\pi(u_kv)) > \ell(\pi(v))$, then

$$
\sum_{t \in \mathbb{F}_q} (\psi \circ f)(t)e_\mu u_1(t_1) \cdots u_k(t_k)v e_\mu = \sum_{t \in \mathbb{F}_q} (\psi \circ (f + \mu_\gamma dc_\gamma y_k))(t)e_\mu u_1(t_1) \cdots u_{k-1}(t_{k-1})u_k v e_\mu.
$$

**Case 2** If $\ell(\pi(u_kv)) < \ell(\pi(v))$, then

$$
\sum_{t \in \mathbb{F}_q} (\psi \circ f)(t)e_\mu u_1(t_1) \cdots u_k(t_k)v e_\mu = \sum_{t \in \mathbb{F}_q, t_k \in \mathbb{F}_q, t_k > t_{k-1} \mu} (\psi \circ (f - \mu_\gamma dy_k^{-1}))(t)e_\mu u_1(t_1) \cdots u_{k-1}(t_{k-1})h_{i_k}(t_k^{-1})v e_\mu,
$$

where $\varphi_k : \mathbb{F}_q[y_1^{\pm 1}, \ldots, y_k^{\pm 1}] \to \mathbb{F}_q[y_1^{\pm 1}, \ldots, y_k^{\pm 1}]$ is given by

$$
\sum_{t \in \mathbb{F}_q, t_k \in \mathbb{F}_q} (\psi \circ f)(t)e_\mu u_1(t_1) \cdots u_{k-1}(t_{k-1})x_{i_k}(t_k^{-1}) = \sum_{t \in \mathbb{F}_q, t_k \in \mathbb{F}_q} (\psi \circ \varphi_k(f))(t)e_\mu u_1(t_1) \cdots u_{k-1}(t_{k-1}).
$$

**Proof.** This Lemma puts $v$ in the place of $u_Tv$ in the proof of Theorem 4.1, (b), and summarizes the steps taken in Case 1 and Case 2. $\blacksquare$
4.3 Global Hecke algebra relations

Fix \( u = u_1u_2 \cdots u_ru_T \in N_\mu \), decomposed according \( s_{i_1}s_{i_2} \cdots s_{i_r} \in W \) (see (2.7)). Suppose \( v' \in N_\mu \) and let \( v = u_Tv' \).

For \( 0 \leq k \leq r \), let \( \tau = (\tau_1, \tau_2, \ldots, \tau_{r-k}) \) be such that \( \tau_i \in \{+0, -0, 1\} \), where \( +0, -0, \) and \( 1 \) are symbols. If \( \tau \) has \( r-k \) elements, then the colength of \( \tau \) is \( \ell(\tau) = k \). For example, if \( r = 10 \) and \( \tau = (-0, 1, +0, +0, 1, 1) \), then \( \ell(\tau) = 4 \). For \( i \in \{+0, -0, 1\} \), let

\[
(i, \tau) = (i, \tau_1, \tau_2, \cdot \cdot \cdot, \tau_{r-k}).
\]

By convention, if \( \ell(\tau) = r \), then \( \tau = \emptyset \).

Suppose \( \ell(\tau) = k \). Define

\[
\Xi^\tau(u, v) = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f^\tau)(t)e_\mu u_1(t_1) \cdots u_k(t_k)v^\tau(t)e_\mu, \tag{4.10}
\]

where

\[
\mathbb{F}_q^r = \left\{ t \in \mathbb{F}_q^r \mid \text{for } k < i \leq r, \text{ if } \tau_{i-k} = +0, \text{ then } t_i \in \mathbb{F}_q^\ast, \right. \nonumber \]

\[
\text{if } \tau_{i-k} = -0, \text{ then } t_i = 0, \nonumber \]

\[
\text{if } \tau_{i-k} = 1, \text{ then } t_i \in \mathbb{F}_q^\ast. \nonumber \]

\[
v^\tau(t) = h_{i_{k+1}}(t_{k+1})^{\tau_1}u_{k+1}^{1-\tau_1} \cdots h_{i_r}(t_r)^{\tau_{r-k}}u_r^{1-\tau_{r-k}}v, \tag{4.12}
\]

with \( +0 = -0 = 0 \in \mathbb{Z}, 1 = 1 \in \mathbb{Z} \) in (4.12); and \( f^\tau \) is defined recursively by

\[
f^0 = f_u = -\frac{\mu_1c_1}{\beta_1(u_T)y_1} - \frac{\mu_2c_2}{\beta_2(u_T)y_2} - \cdots - \frac{\mu_rc_r}{\beta_r(u_T)y_r}, \quad \text{(as in (4.7))}, \tag{4.13}
\]

\[
f(i, \tau) = \begin{cases} f^\tau + \mu_\gamma d_\tau y_k, & \text{if } i = \pm 0, \\ \varphi_k(f^\tau) - \mu_\gamma d_\tau y_k^{-1}, & \text{if } i = 1, \end{cases} \tag{4.14}
\]

where \((v^\tau)^{-1}x_{\alpha_{ik}}(t)v^\tau = x_{\gamma_r}(d_\tau t)\) and the map \( \varphi_k \) is as in Lemma 4.2, Case 2.

Remarks.

1. By (4.10) and Theorem 4.1 (a), \( \Xi^0(u, v) = (e_\mu u e_\mu)(e_\mu v' e_\mu) \) (recall, \( v = u_Tv' \)).

2. If \( \ell(\tau) = 0 \) so that \( \tau \) is a string of length \( r \), then

\[
\begin{align*}
\Xi^\tau(u, v) &= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f^\tau)(t)e_\mu v^\tau e_\mu \text{ has no remaining factors of the form } u_k(t_k), \\
\Xi^\tau(u, v) &= 0 \text{ unless } v^\tau(t) \in N_\mu \text{ for some } t \in \mathbb{F}_q.
\end{align*}
\]

The following corollary gives relations for expanding \( \Xi^\tau(u, v) \) (beginning with \( \Xi^0(u, v) \)) as a sum of terms of the form \( \Xi^\tau' \) with \( \ell(\tau') = \ell(\tau) - 1 \). When each term has colength 0 (length \( r \)), then the product \((e_\mu u e_\mu)(e_\mu v' e_\mu)\) is decomposed in terms of the basis elements of \( H_\mu \).

In summary, while we compute \( f^\tau \) recursively by removing elements from \( \tau \), we compute the product \((e_\mu u e_\mu)(e_\mu v' e_\mu)\) by progressively adding elements to \( \tau \).

**Corollary 2 (The Global Alternative).** Let \( u, v' \in N_\mu \) such that \( u = u_1u_2 \cdots u_ru_T \) decomposes according to a minimal expression in \( W \). Let \( v = u_Tv' \). Then

\[
\Xi^0(u, v),
\]

(a) \((e_\mu u e_\mu)(e_\mu v' e_\mu) = \Xi^0(u, v),\)
(b) If $\ell'(\tau) = k$, then

$$
\Xi^\tau(u,v) = \begin{cases} 
\Xi^{(+0,\tau)}(u,v), & \text{if } \ell(\pi(u_kv^r)) > \ell(\pi(v^r)), \\
\Xi^{(-0,\tau)}(u,v) + \Xi^{(1,\tau)}(u,v), & \text{if } \ell(\pi(u_kv^r)) < \ell(\pi(v^r)). 
\end{cases}
$$

Proof. (a) follows from Remark 1. (b) Suppose $\ell'(\tau) = k$. Note that

$$
\Xi^\tau(u,v) = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f^\tau)(t) e_\mu u_1(t_1) \ldots u_k(t_k)v^r e_\mu 
$$

$$
= \frac{1}{q^r} \sum_{t' \in (\mathbb{F}_q^{r-k})^r} \sum_{t'' \in \mathbb{F}_q^k} (\psi \circ f^\tau)(t', t'') e_\mu u_1(t_1) \ldots u_k(t_k)v^r e_\mu 
$$

where $(\mathbb{F}_q^{r-k})^r = \{ (t_{k+1}, \ldots, t_r) \in \mathbb{F}_q^{r-k} \mid \text{restrictions according to } \tau \}$ (as in (4.11)). Apply Lemma 4.2 to the inside sum with $f := f^r$, $v := v^r$. Note that the Lemma relations imply

$$
\{ t' \in \mathbb{F}_q^k \} \text{ becomes } \begin{cases} 
\{ t' \in \mathbb{F}_q^k \}, & \text{if in Case 1}, \\
\{ t' \in \mathbb{F}_q^k \mid t_k = 0 \}, & \text{if in Case 2, first sum}, \\
\{ t' \in \mathbb{F}_q^k \mid t_k \in \mathbb{F}_q^k \}, & \text{if in Case 2, second sum}, 
\end{cases}
$$

$$
f^\tau \text{ becomes } \begin{cases} 
f^{(+0,\tau)}, & \text{if in Case 1}, \\
f^{(-0,\tau)}, & \text{if in Case 2, first sum}, \\
f^{(1,\tau)}, & \text{if in Case 2, second sum}. 
\end{cases}
$$

$$
v^\tau \text{ becomes } \begin{cases} 
v^{(+0,\tau)}, & \text{if in Case 1}, \\
v^{(-0,\tau)}, & \text{if in Case 2, first sum}, \\
v^{(1,\tau)}, & \text{if in Case 2, second sum}. 
\end{cases}
$$

Thus,

$$
\Xi^\tau(u,v) = \begin{cases} 
\Xi^{(+0,\tau)}(u,v), & \text{if Case 1}, \\
\Xi^{(-0,\tau)}(u,v) + \Xi^{(1,\tau)}(u,v), & \text{if Case 2}, 
\end{cases}
$$

as desired. \hfill \Box

5 The case $G = GL_n(\mathbb{F}_q)$

Let $G = GL_n(\mathbb{F}_q)$ be the general linear group over the finite field $\mathbb{F}_q$ with $q$ elements. Define the subgroups

$$
T = \left\{ \begin{array}{c} \text{diagonal matrices} \\
\text{monomial matrices} \end{array} \right\}, \quad N = \left\{ \begin{array}{c} \text{diagonal matrices} \\
\text{monomial matrices} \end{array} \right\}, 
$$

$$
W = \left\{ \begin{array}{c} \text{permutation matrices} \\
\text{permutation matrices} \end{array} \right\}, \quad \text{and } U = \left\{ \begin{array}{c} \begin{pmatrix} 1 & \cdots & * \\
0 & \cdots & 1 \end{pmatrix} \end{array} \right\}, 
$$

where a monomial matrix is a matrix with exactly one nonzero entry in each row and column.

Let $x_{ij}(t) \in U$ be the matrix with $t$ in position $(i,j)$, ones on the diagonal and zeroes elsewhere; write $x_i(t) = x_{i,i+1}(t)$. Let $h_{\epsilon_i}(t) \in T$ denote the diagonal matrix with $t$ in the $i$th slot and ones elsewhere, and let $s_i \in W \subseteq N$ be the identity matrix with the $i$th and $(i+1)$st columns interchanged. That is,

$$
x_i(t) = Id_{i-1} \oplus (\begin{smallmatrix} 1 & \vdots \\
0 & \ddots & 1 \end{smallmatrix}) \oplus Id_{n-i-1}, \quad h_{\epsilon_i}(t) = Id_{i-1} \oplus (t) \oplus Id_{n-i}, \\
s_i = Id_{i-1} \oplus (\begin{smallmatrix} 1 & \vdots \\
0 & \ddots & 1 \end{smallmatrix}) \oplus Id_{n-i-1}, \quad (5.2)
$$
where $Id_k$ is the $k \times k$ identity matrix. Then

$$W = \langle s_1, s_2, \ldots, s_{n-1} \rangle, \quad T = \langle h_{\varepsilon_i}(t) \mid 1 \leq i \leq n, t \in \mathbb{F}_q^* \rangle, \quad N = WT,$$

$$U = \langle x_{ij}(t) \mid 1 \leq i < j \leq n, t \in \mathbb{F}_q \rangle, \quad G = \langle U, W, T \rangle. \quad (5.3)$$

The Chevalley group relations for $G$ are (see also Section 4.1)

$$x_{ij}(a)x_{rs}(b) = x_{ij}(b)x_{rs}(a)x_{is}(\delta_{jr}ab)x_{rj}(-\delta_{is}ab), \quad (i, j) \neq (r, s), \quad (U1)$$

$$x_{ij}(a)x_{ij}(b) = x_{ij}(a + b), \quad (U2)$$

$$s_i^2 = 1, \quad (N1)$$

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad \text{and} \quad s_is_j = s_js_i, \quad |i - j| > 1, \quad (N2)$$

$$h_{\varepsilon_i}(b)h_{\varepsilon_j}(a) = h_{\varepsilon_j}(b)h_{\varepsilon_i}(a), \quad (N3)$$

$$s_i x_{ij}(t) = x_{s_i(i)s_r(j)}(t)s_r, \quad (UN1)$$

$$x_{ij}(a)h_{\varepsilon_r}(t) = h_{\varepsilon_r}(t)x_{ij}(t^{-\delta_{ir}}t^{\delta_{ir}}a), \quad (UN2)$$

$$s_is_j(t)s_t = x_i(t^{-1})s_is_j(-t)h_{\varepsilon_i}(t)h_{\varepsilon_{i+1}}(-t^{-1}), \quad t \neq 0, \quad (UN3)$$

where $\delta_{ij}$ is the Kronecker delta.

### 5.1 A pictorial version of $GL_n(\mathbb{F}_q)$

For the results that follow, it will be useful to view these elements of $CG$ as braid-like diagrams instead of matrices. Consider the following depictions of elements by diagrams with vertices, strands between the vertices, and various objects that slide around on the strands. View

$$s_i \quad \text{as} \quad \begin{array}{cccc} \cdots & & \cdots \\ \cdots & & \cdots \\ \cdots & & \cdots \\ \cdots & & \cdots \end{array}, \quad (5.4)$$

$$h_{\varepsilon_i}(t) \quad \text{as} \quad \begin{array}{cccc} \cdots & & \cdots & & \cdots \\ \cdots & & \cdots & & \cdots \\ \cdots & & \cdots & & \cdots \end{array}, \quad (5.5)$$

$$x_{ij}(ab) \quad \text{as} \quad \begin{array}{cccc} \cdots & & \cdots & & \cdots \\ \cdots & & \cdots & & \cdots \\ \cdots & & \cdots & & \cdots \end{array}, \quad (5.6)$$

where each diagram has two rows of $n$ vertices. Multiplication in $G$ corresponds to the concatenation of two diagrams; for example, $s_2s_1$ is

$$s_2s_1 = \begin{array}{cccc} \cdots & & \cdots & & \cdots \\ \cdots & & \cdots & & \cdots \\ \cdots & & \cdots & & \cdots \end{array}.$$

In the following Chevalley relations, curved strands indicate longer strands, so for example (UN1) indicates that $\overrightarrow{\delta}$ and $\overleftarrow{\delta}$ slide along the strands they are on (no matter how long). The
Chevalley relations are

\[
\begin{align*}
\begin{array}{cccc}
\begin{array}{c}
\bullet \bullet \\
\text{a} \text{b}
\end{array} & \text{\rightarrow} & \begin{array}{c}
\bullet \\
\text{1}
\end{array} & \text{\rightarrow} & \begin{array}{c}
\bullet \\
\text{a+b}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
&= \begin{array}{c}
\bullet \\
\text{1}
\end{array} & \text{\rightarrow} & \begin{array}{c}
\bullet \\
\text{a}
\end{array} & \text{\rightarrow} & \begin{array}{c}
\bullet \\
\text{b}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{cccc}
\begin{array}{c}
\text{a} \text{b}
\end{array} & \text{\leftarrow} & \begin{array}{c}
\text{1}
\end{array} & \text{\leftarrow} & \begin{array}{c}
\text{a+b}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
&\begin{array}{c}
\bullet \\
\text{a}
\end{array} & \text{\rightarrow} & \begin{array}{c}
\bullet \\
\text{b}
\end{array} \begin{array}{c}
\bullet \\
\text{b}
\end{array} & \begin{array}{c}
\bullet \\
\text{b}
\end{array} & \begin{array}{c}
\bullet \\
\text{1}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{cccc}
\begin{array}{c}
\text{a} \text{b}
\end{array} & \text{\leftarrow} & \begin{array}{c}
\text{1}
\end{array} & \text{\leftarrow} & \begin{array}{c}
\text{a+b}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
&= \begin{array}{c}
\bullet \\
\text{1}
\end{array} & \text{\rightarrow} & \begin{array}{c}
\bullet \\
\text{a}
\end{array} & \text{\rightarrow} & \begin{array}{c}
\bullet \\
\text{b}
\end{array}
\end{align*}
\]

5.2 The unipotent Hecke algebra $H_\mu$

Fix a nontrivial group homomorphism $\psi : \mathbb{F}_q^+ \rightarrow \mathbb{C}^*$, and a map

\[
\mu : \{(i,j) \mid 1 \leq i < j \leq n\} \rightarrow \mathbb{F}_q \quad \text{with} \quad \mu_{ij} = 0 \text{ for } j \neq i + 1.
\] (5.7)

Then

\[
\psi_\mu : \quad \begin{array}{c}
U \\
\mu_{ij}
\end{array} \rightarrow \mathbb{C}^* \\
\begin{array}{c}
x_{ij}(t)
\end{array} \mapsto \psi(\mu_{ij}t)
\] (5.8)

is a group homomorphism. Since $\mu_{ij} = 0$ for all $j \neq i + 1$, write

\[
\mu = (\mu(1), \mu(2), \ldots, \mu(n-1), \mu(n)), \quad \text{where} \quad \mu(i) = \mu_{i,i+1} \text{ and } \mu(n) = 0.
\] (5.7)

In fact, we may assume that $\mu_{i(i)} \in \{0, 1\}$ for all $1 \leq i \leq n$.

The unipotent Hecke algebra $\mathcal{H}_\mu$ of the triple $(G,U,\psi_\mu)$ is

\[
\mathcal{H}_\mu = \text{End}_G(\text{Ind}_U^G(\psi_\mu)) \cong e_\mu CGe_\mu, \quad \text{where} \quad e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1})u.
\] (5.9)

If $N_\mu = \{v \in N \mid e_\mu ve_\mu \neq 0\}$, then $\{e_\mu ve_\mu \mid v \in N_\mu\}$ is a basis for $\mathcal{H}_\mu$. 

18
We may characterize the elements of $N_\mu$ in the following fashion (for a more extensive analysis of $N_\mu$ see [Th]). Suppose $v \in N$. For each $\mu(i) = 0$, place a dotted line between the $i$th and $(i+1)$st vertices; for example, $\mu = (1, 0, 1, 1, 0, 0)$ gives

Then $e_\mu v e_\mu \neq 0$ if and only if the diagram for $v$ satisfies

(1) if \[
\begin{array}{cc}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bullet & \bullet \\
\end{array}
\]
adjacent, then

(2) if \[
\begin{array}{cc}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bullet & \bullet \\
\end{array}
\]
then

(3) if \[
\begin{array}{cc}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bullet & \bullet \\
\end{array}
\]
then either \[
\begin{array}{cc}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bullet & \bullet \\
\end{array}
\]
or \[
\begin{array}{cc}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bullet & \bullet \\
\end{array}
\]

Example. If $\mu = (1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1)$ then

5.3 Pictorial versions of $e_\mu v e_\mu$

Note that the map

$$\pi : N = WT \longrightarrow W$$

$$w h \mapsto w, \text{ for } w \in W, h \in T,$$

is a surjective group homomorphism. Since $W \subseteq N$, adjust the decomposition of (2.7) as follows. Let $u \in N$ with $\pi(u) = s_{i_1} \cdots s_{i_r}$ for $r$ minimal. Then $u$ has a unique decomposition

$$u = u_1 u_2 \ldots u_r u_T, \text{ where } u_k = s_{i_k}, u_T \in T.$$ (5.11)

For $t \in \mathbb{F}_q$, write $u_k(t) = s_{i_k} x_{i_k}(t)$.

Fix a composition $\mu$. The decomposition

$$U = \prod_{1 \leq i < j \leq n} U_{ij} \text{ where } U_{ij} = \langle x_{ij}(t) \mid t \in \mathbb{F}_q \rangle,$$

19
implies
\[ e_\mu = \prod_{1 \leq i < j \leq n} e_{ij}(1), \quad \text{where} \quad e_{ij}(k) = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\mu_{ij}kt)x_{ij}(t). \quad (5.12) \]

Pictorially, let
\[ u_k \text{ as } \quad \begin{array}{c}
\vdots \\
\bullet \\
\vdots
\end{array} \quad (5.13) \]
\[ u_k(t_k) \text{ as } \quad \begin{array}{c}
\vdots \\
\bullet \\
\vdots
\end{array} \quad (5.14) \]
\[ e_{ij}(k) \text{ as } \quad \begin{array}{c}
\vdots \\
\bullet \\
\vdots
\end{array} \quad (5.15) \]
\[ e_\mu \text{ as } \quad \begin{array}{c}
\mu(1) \\
\mu(2) \\
\mu(n-1)
\end{array} \quad (5.16) \]

Examples:
1. If \( u = u_1u_2 \cdots u_8u_T \in N \) decomposes according to \( s_3s_1s_2s_3s_1s_4s_2s_3 \in W \) with \( u_T = \text{diag}(a, b, c, d, e) \), then

\[ u = \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \quad , \quad u_1(t_1)u_2(t_2) \cdots u_8(t_8)u_T = \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \quad \]

and

\[ e_\mu u e_\mu = \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array} \quad . \]
2. If \( n = 5 \), then (5.12) implies

\[
(e_\mu = e_{45}(1)e_{35}(1)e_{34}(1)e_{25}(1)e_{24}(1)e_{23}(1)e_{15}(1)e_{14}(1)e_{13}(1)e_{12}(1)).
\]

The elements \( e_{ij}(k) \) also interact with \( U \) and \( N \) as follows (see also Section 4.1)

\[
s_r e_{ij}(k) s_r = e_{s_r(i)s_r(j)}(k),
\]

\[
e_{\mu} e_{ij}(1) = e_{\mu} v, \quad v \in N_\mu, (\pi v)(i) < (\pi v)(j), \tag{E1}
\]

\[
e_{ij}(k) h_{ij}(r) = h_{ij}(r) e_{ij}(k r^\delta u_r - \delta u), \tag{E2}
\]

\[
e_{\mu} x_{ij}(t) = \psi(\mu_{ij} t) e_{\mu} = x_{ij}(t) e_{\mu}, \tag{E3}
\]

or pictorially,

\[
\begin{align*}
  \includegraphics[width=0.2\textwidth]{image1.png} & = \includegraphics[width=0.2\textwidth]{image2.png} \tag{E1} \\
  \includegraphics[width=0.2\textwidth]{image3.png} & = \includegraphics[width=0.2\textwidth]{image4.png} \tag{E2} \\
  \includegraphics[width=0.2\textwidth]{image5.png} & = \includegraphics[width=0.2\textwidth]{image6.png} \tag{E3} \\
  \includegraphics[width=0.2\textwidth]{image7.png} & = \includegraphics[width=0.2\textwidth]{image8.png} \tag{E4}
\end{align*}
\]

and for \( v \in N_\mu \),

\[
\includegraphics[width=0.4\textwidth]{image9.png} = \includegraphics[width=0.4\textwidth]{image10.png} \tag{E2}
\]

Suppose \( u = u_1 u_2 \cdots u_r u_T \in N_\mu \) with \( u_T = \text{diag}(h_1, h_2, \ldots, h_n) \). Then using (5.12), (E3), (E1) and (E2), we can rewrite \( e_{\mu} ue_{\mu} \) as

\[
\begin{align*}
  \includegraphics[width=0.5\textwidth]{image11.png} & = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f_u)(t) \\
  \includegraphics[width=0.5\textwidth]{image12.png} & = u_1(t_1) u_2(t_2) \cdots u_r(t_r) \tag{5.17}
\end{align*}
\]

where \( f_u \in \mathbb{F}_q[y_1, y_2, \ldots, y_r] \) is given by

\[
f_u(y_1, y_2, \ldots, y_r) = -\mu_{i_1j_1} h_{i_1} h_{j_1}^{-1} y_1 - \mu_{i_2j_2} h_{i_2} h_{j_2}^{-1} y_2 - \cdots - \mu_{i_rj_r} h_{i_r} h_{j_r}^{-1} y_r, \tag{5.18}
\]
for \((i_k,j_k) = (a,b)\), if the \(k\)th crossing in \(u\) crosses the strands coming from the \(a\)th and \(b\)th top vertices.

**Example (continued).** Suppose \(u = u_1 u_2 \cdots u_8 u_T \in N\) decomposes according to \(s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3 \in W\) and \(u_T = \text{diag}(a, b, c, d, e) \in T\) (as in Example 1 above). Consider the ordering

\[
e^\mu = \varepsilon \sum_{i \in \mathbb{E}^{10}_8} (\psi \circ f)(t) u_1 x_{34}(\frac{a}{b}) t_1 u_2 x_{12}(\frac{d}{e}) t_2 u_3 x_{23}(\frac{a}{d}) t_3 \cdots u_8 x_{34}(\frac{c}{d}) u_T,
\]

where by (5.12), \(f = -y_1 - y_2 - y_8\). Therefore, by renormalizing

\[
e^\mu u e^\mu = \frac{e^\mu}{q^8} \sum_{t' \in \mathbb{P}^3_q} (\psi \circ f_u)(t') u_1 x_{34}(t_1) u_2 x_{12}(t_2) u_3 x_{23}(t_3) \cdots u_8 x_{34}(t_8) u_T
\]

where \(f_u = -ba^{-1}y_1 - e^{-1}y_2 - dc^{-1}y_8\). Pictorially, by sliding the \(e_{ij}(1)\) down along the strands until they get stuck, this computation gives

\[
e^\mu u e^\mu = \frac{1}{q^8} \sum_{t' \in \mathbb{P}^3_q} (\psi \circ f_u)(t)
\]

where \(f_u = -ba^{-1}y_1 - e^{-1}y_2 - dc^{-1}y_8\) (as in (5.18)), since \((i_1,j_1) = (1,2)\), \((i_2,j_2) = (4,5)\), \((i_3,j_3) = (1,5)\), etc.
5.4 Relations for multiplying basis elements.

Let \( u = u_1 u_2 \cdots u_r u_T \in N_\mu \) decompose according to a minimal expression in \( W \) as in (5.11). Let \( v \in N_\mu \) and use (N3) and (N4) to write \( u_T v = w \cdot \text{diag}(a_1, a_2, \cdots, a_n) \) for some \( w = \pi(v) \in W \) (see (5.7)). Then use (5.17) to write

\[
(e_\mu u e_\mu)(e_\mu v e_\mu) = \frac{1}{q^r} \sum_{t \in F_{q^r}} (\psi \circ f_u)(t) u_1(t_1) u_2(t_2) \cdots u_r(t_r) \]

(5.20)

(This form corresponds to \( \Xi^\emptyset(u, u_T v) \) of Corollary 2).

Example (continued). If \( u \) is as in (5.19) and \( v = s_2 s_3 s_2 s_1 s_2 \cdot \text{diag}(f, g, h, i, j) \in N \), then

\[
(e_\mu u e_\mu)(e_\mu v e_\mu) = \frac{1}{q^3} \sum_{t \in F_{q^3}} (\psi \circ f_u)(t)
\]

Consider the crossing in (5.20) corresponding to \( u_r(t_r) \). There are two possibilities.

Case 1 the strands that cross at \( (\overline{7}) \) do not cross again as they go up to the top of the diagram \((\ell(u_r w) > \ell(w))\),

Case 2 the strands that cross at \( (\overline{3}) \) cross once on the way up to the top of the diagram \((\ell(u_r w) < \ell(w))\).

In the first case, by (UN1), (UN2) and (E4),

\[
e_\mu u e_\mu v e_\mu = \frac{1}{q} \sum_{t \in F_{q}} (\psi \circ f_u)(t)
\]
where $f^{(+0)} = f_u + \mu_i a_j a_i^{-1} y_r$.

In the second case,

$$e_{t_r} e_{t_r} = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q} (\psi \circ f_u)(t)$$

Use (UN3) and (N1) to split the sum into two parts corresponding to $t_r = 0$ and $t_r \neq 0$, 

$$e_{t_r} e_{t_r} = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q} (\psi \circ f_u)(t)$$
where \( f^{(-0)} = f_u \). Now use (UN1), (UN2), (U2), (U1) and (E4) on the second sum to get

\[
e_{\mu} u_{\mu} v e_{\mu} = \frac{1}{q} \sum_{\mathbf{t} \in F} (\psi \circ f^{(-0)})(t)
\]

\[
+ \frac{1}{q} \sum_{\mathbf{t} \in F} (\psi \circ f^{(1)})(t)
\]

where \( f^{(1)} = \varphi_r(f_u) + \mu_{ij} a_j a_i^{-1} y_r^{-1} \), and \( \varphi_r(f) \) is defined by

\[
\sum_{\mathbf{t} \in F} (\psi \circ f)(t) = \sum_{\mathbf{t} \in F} (\psi \circ \varphi_r(f))(t)
\]

Remarks:

(a) We could have applied these steps for any \( f, u, \) and \( v \), so we can iterate the process with each sum.

(b) The most complex step in these computations is determining \( \varphi_r \). The following section will develop an efficient algorithm for computing the right-hand side of (*).

5.5 Computing \( \varphi_k \) via painting, paths and sinks.

Painting algorithm \((u^\circ)\). Suppose \( u = u_1 u_2 \cdots u_r \in N \) decomposes according to \( s_{i_1} s_{i_2} \cdots s_{i_k} \in W \) (assume \( u_T = 1 \)). Paint flows down strands (by gravity). Each step is illustrated with the example \( u = u_1 u_2 \cdots u_8 \), decomposed according to \( s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3 \in W \).

1) Paint the left [respectively right] strand exiting \( k \) red [blue] all the way to the bottom of the diagram.

where red is , blue is , and \( k \) is .
(2) For each crossing that the red [blue] strand passes through, paint the right [left] strand (if possible) red [blue] until that strand either reaches the bottom or crosses the blue [red] strand of (1).

Let
\[ u^\otimes = u_1(t_1)u_2(t_2) \cdots u_k(t_k) \] painted according to the above algorithm \hspace{1cm} (5.24)

**Sinks and paths.** The diagram \( u^\otimes \) has a *crossed sink* at \( j \) if \( j \) is a crossing between a red strand and a blue one, or

\[ \bullet \]

Note that since \( u \) is decomposed according to a minimal expression in \( W \), there will be no crossings of the form

\[ \bullet \]

The diagram \( u^\otimes \) has a *bottom sink* at \( j \) if a red strand enters \( j \)th bottom vertex and a blue strand enters the \((j + 1)\)st bottom vertex, or

\[ \bullet \]

A *red [respectively blue]* path \( p \) from a sink \( s \) (either crossed or bottom) in \( u^\otimes \) is an increasing sequence

\[ j_1 < j_2 < \cdots < j_l = k, \]

such that in \( u^\otimes \)

(a) \( j_m \) is directly connected (no intervening crossings) to \( j_{m+1} \) by a red [blue] strand,

(b) if \( s \) is a crossed sink, then \( j_1 = s \),

(b') if \( s \) is a bottom sink, then

- in a red path, the \( s \)th bottom vertex connects to the crossing \( j_1 \) with a red strand.
- in a blue path, the \((s + 1)\)st bottom vertex connects to the crossing \( j_1 \) with a blue strand.
Let
\[
P_\circ(u_\circ, s) = \begin{cases} 
\text{red paths from } s \text{ in } u_\circ & \\
\text{blue paths from } s \text{ in } u_\circ
\end{cases}
\]
and
\[
P_\circ'(u_\circ, s) = \begin{cases} 
\text{blue paths from } s \text{ in } u_\circ & \\
\text{red paths from } s \text{ in } u_\circ
\end{cases}
\] (5.25)

The weight of a path \( p \) is
\[
\text{wt}(p) = \begin{cases} 
\prod_{p \text{ switches strands at } i} y_i, & \text{if } p \in P_\circ(u_\circ, s), \\
\prod_{p \text{ switches strands at } i} (-y_i), & \text{if } p \in P_\circ'(u_\circ, s).
\end{cases}
\] (5.26)

Each sink \( s \) in \( u_\circ \) (either crossed \( j \) or bottom \( j \)) has an associated polynomial \( g_s \in F_q[y_1, y_2, \ldots, y_{k-1}, y_{k}^{-1}] \) given by
\[
g_s = \sum_{p \in P_\circ(u_\circ, s) \atop p' \in P_\circ'(u_\circ, s)} \text{wt}(p)y_k^{-1}\text{wt}(p').
\] (5.27)

Example (continued). If \( u = u_1 u_2 \cdots u_8 \) decomposes according to \( s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3 \) (as in (5.23)), then

\[
\begin{array}{c}
P_\circ(u_\circ, 4) = \begin{cases} 
\begin{array}{c}
7
\end{array}, & \\
\begin{array}{c}
5
\end{array}, & \\
\begin{array}{c}
1
\end{array}, &
\end{cases}
\end{array}
\quad
\begin{array}{c}
P_\circ'(u_\circ, 4) = \begin{cases} 
\begin{array}{c}
4
\end{array}, & \\
\begin{array}{c}
8
\end{array}, &
\end{cases}
\end{array}
\]
\]

with \( \text{wt}(1 < 3 < 5 < 7 < 8) = y_5 \), \( \text{wt}(1 < 4 < 7 < 8) = y_1 y_7 \), and \( \text{wt}(6 < 8) = 1 \). The corresponding polynomial is
\[
g_4 = y_5 y_8^{-1} + y_1 y_7 y_8^{-1}.
\] (5.28)

Lemma 5.1. Let \( u = u_1 u_2 \cdots u_r \) and \( \varphi_r \) be as in (5.22) and (\ast); suppose \( u_\circ \) is painted as above. Then
\[
\varphi_r(f) = f \bigg|_{\{y_j \rightarrow y_j - g_\circ \mid \circ \text{ a crossed sink}\}} + \sum_{j \text{ a bottom sink}} \mu(j)g_j.
\]

Proof. In the painting,

marks a strand travelled by

and

marks a strand travelled by

Substitutions due to crossed sinks correspond to the normalizations in relation (U2), and the sum over bottom sinks comes from applications of relation (E4). \( \square \)
Example (continued) Recall \( u = s_3s_1s_2s_3s_1s_4s_2s_3 \). Then \( u^\otimes \) has crossed sinks at \( 2 \), \( 3 \), and \( 4 \). The only bottom sink is at 4. Therefore,

\[
\varphi_8(f) = f \bigg|_{y_4 = y_4^{-1}y_2^{-1}} + \mu(4)g_4 = f \bigg|_{y_4 = y_4^{-1}y_4^{-1}y_6 + y_5y_8^{-1}y_1y_7} + \mu(4)(y_5y_8^{-1} + y_1y_7y_8^{-1}).
\]

(for example, \( g_4 \) was computed in (5.28)).

5.6 A multiplication algorithm

**Theorem 5.1 (The algorithm).** Let \( G = GL_n(\mathbb{F}_q) \) and \( u, v \in N_\mu \). An algorithm for multiplying \( e_\mu u e_\mu \) and \( \epsilon_\mu v e_\mu \) is

1. Decompose \( u = u_1u_2 \cdots u_n \) according to some minimal expression in \( W \) (as in (5.11)).
2. Put \( e_\mu u e_\mu \) into the form specified by (5.20), with \( u_T = w \cdot \text{diag}(a_1, a_2, \ldots, a_n) \) (\( w = \pi(v) \in W \)).
3. Complete the following
   
   (a) If \( \ell(u_r, w) > \ell(w) \), then apply relation (5.21).
   (b) If \( \ell(u_r, w) < \ell(w) \), then apply relation (5.22), using (5.11) to compute \( \varphi_r \).
4. If \( r > 1 \), then reapply (3) to each sum with \( r := r - 1 \) and with
   
   (a) \( w := u_r, w \), after using (3a) or using (3b), in the first sum,
   (b) \( w := w \), after using (3b), in the second sum.
5. Set all diagrams not in \( N_\mu \) to zero.

**Sample computation.** Suppose \( n = 3 \) and \( \mu(i) = 1 \) for all \( 1 \leq i \leq 3 \) (i.e., the Gelfand-Graev case). Then

\[
N_\mu = \left\{ \begin{array}{c} a \ a \ a \ b \ a \ a \ a \ b \ b \ b \ c \ \ a, b, c \in \mathbb{F}_q \end{array} \right\}
\]

Suppose

\[
u = \begin{array}{c} a \ b \ c \ \ \ \ \ \ \ \ \ \ d \ e \ e \ \ \ \ \ \end{array} \]

1. **Theorem 5.1 (1):** Let \( u = u_1u_2u_3u_T \in N_\mu \) decompose according to \( s_2s_1s_2 \in W \), with \( u_T = \text{diag}(a, b, c) \).
2. **Theorem 5.1 (2):** By (5.20)
with \( uTv = s_2s_1 \cdot \text{diag}(cd, ae, be) \) (so \( w = s_2s_1 \)), and \( f_u = -\frac{b}{a} y_1 - \frac{c}{b} y_3 \) (as in (5.18)).

3. **Theorem 5.1 (3b):** Since \( \ell(u_3w) < \ell(w) \), paint \( u_1(t_1)u_2(t_2)u_3(t_3) \) to get \((u_1u_2u_3)\oplus\) (as in (5.24)),

\[
\frac{1}{q^3} \sum_{t \in \mathbb{F}_q^3} (\psi \circ f_u)(t)
\]

Now apply (5.22),

\[
\frac{1}{q^3} \sum_{t \in \mathbb{F}_q^3} (\psi \circ f^{(-0)})(t) + \frac{1}{q^3} \sum_{t \in \mathbb{F}_q^3} (\psi \circ f^{(1)})(t)
\]

where \( f^{(-0)} = -\frac{b}{a} y_1 - \frac{c}{b} y_3 \) and by Lemma 5.1,

\[
f^{(1)} = \varphi_3(f_u) + \mu_{13} \frac{be}{cd} y_3^{-1} = -\frac{b}{a} y_1 + \frac{b}{a} y_2 y_3^{-1} - \frac{c}{b} y_3 + y_3^{-1}.
\]

4. **Theorem 5.1 (4):** Set \( r := 2 \) with \( w := u_r w = s_1 \) in the first sum and \( w := w \) in the second sum.

5. **Theorem 5.1 (3a) (3b):** In the first sum, \( \ell(u_2s_1) < \ell(s_1) \), so paint \( u_1(t_1)u_2(t_2) \) to get \((u_1u_2)\oplus\). In the second sum, \( \ell(u_2s_2s_1) > \ell(s_2s_1) \), so apply (5.21),

\[
\frac{1}{q^3} \sum_{t \in \mathbb{F}_q^3} (\psi \circ f^{(-0)})(t) + \frac{1}{q^3} \sum_{t \in \mathbb{F}_q^3} (\psi \circ f^{(+0,1)})(t)
\]

where \( f^{(+0,1)} = -\frac{b}{a} y_1 + \frac{b}{a} y_2 y_3^{-1} - \frac{c}{b} y_3 + y_3^{-1} - \mu(3) \frac{b}{a} y_2 y_3^{-1} = -\frac{b}{a} y_1 - \frac{c}{b} y_3 + y_3^{-1} \). Now apply (5.22) to the first sum,
where \( f(-0,-0) = -\frac{b}{a}y_1 - \frac{c}{b} y_3 \) and \( f(1,-0) = \varphi(f(-0)) + \mu(1) \frac{ac}{cd} y_2^{-1} = -\frac{b}{a} y_1 - y_2^{-1} y_1 + \frac{ac}{cd} y_2^{-1} \).

6. **Theorem 5.1 (4):** Set \( r = 1 \) with \( w := u_2 s_1 = 1 \) in the first sum, \( w := s_1 \) in the second sum, and \( w := s_1 s_2 s_1 \) in the third sum.

7. **Theorem 5.1 (3a) (3a) (3b):** In the first sum \( \ell(s_2 1) > \ell(1) \), so apply (5.21); in the second sum \( \ell(s_2 s_1) > \ell(s_1) \), so apply (5.21); in the third sum, \( \ell(s_2 s_1 s_2 s_1) < \ell(s_1 s_2 s_1) \), so paint \( u_1(t_1) \) to get \( u_1^{(3)} \).

where \( f(0,0,-0) = -\frac{b}{a} y_1 - \frac{c}{b} y_3 + \frac{b}{a} y_1 = -\frac{c}{b} y_3 \) and \( f(0,1,0) = -\frac{b}{a} y_1 + \frac{a}{cd} y_2^{-1} y_2^{-1} y_1 \). Now apply (5.22) to the third sum

where \( f(0,0,1) = -\frac{c}{b} y_3 + y_3^{-1} \)

\[ f(1,0,1) = \varphi_1(f(0,1)) + \mu(1) \frac{ae}{cd} y_1^{-1} y_3^{-1} = -\frac{b}{a} y_1 - \frac{c}{b} y_3 + y_3^{-1} y_1^{-1} + \frac{ae}{cd} y_1^{-1} y_3^{-1}. \]

8. **Theorem 5.1 (5):** The first sum contains no elements of \( N_\mu \), so set it to zero. The second sum contains elements of \( N_\mu \) when \( be = -ae t_2^{-1} \), so set \( t_2 = -\frac{a}{b} \). The third sum contains
elements of $N_\mu$ when $cdt_3 = ae$, so set $t_3 = \frac{ae}{cd}$. All the terms in the fourth sum are in $N_\mu$.

$$= 0 + \frac{1}{q^3} \sum_{t \in F_3^*} (\psi \circ f^{(+0,1,-0)}(t))$$

$$+ \frac{1}{q^3} \sum_{t_1, t_3 = \frac{ae}{cd}} (\psi \circ f^{(-0,+0,1)}(t)) + \frac{1}{q^3} \sum_{t_1, t_3 = \frac{ae}{cd}} (\psi \circ f^{(+1,1,0)}(t))$$

$$= \frac{1}{q^2} \psi\left(-\frac{be}{cd}\right) + \frac{1}{q^2} \psi\left(-\frac{ae}{bd} + \frac{cd}{ae}\right)$$

$$+ \frac{1}{q^2} \sum_{t_1, t_3} \psi\left(-\frac{b}{a} t_1 - \frac{c}{b} t_3 + t_3^{-1} + t_1^{-1} + \frac{ae}{cd} t_1^{-1} t_3^{-1}\right).$$

References


[St] Steinberg, R. *Lectures on Chevalley groups.* mimeographed notes, Yale University, 1967.


