Unipotent Hecke algebras of $GL_n(\mathbb{F}_q)$

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Abstract

This paper describes a family of Hecke algebras $\mathcal{H}_\mu = \text{End}_G(\text{Ind}_U^G(\psi_\mu))$, where $U$ is the subgroup of unipotent upper-triangular matrices of $G = GL_n(\mathbb{F}_q)$ and $\psi_\mu$ is a linear character of $U$. The main results combinatorially index a basis of $\mathcal{H}_\mu$, provide a large commutative subalgebra of $\mathcal{H}_\mu$, and after describing the combinatorics associated with the representation theory of $\mathcal{H}_\mu$, generalize the RSK correspondence that is typically found in the representation theory of the symmetric group.

1 Introduction

Iwahori [Iw] and Iwahori-Matsumoto [IM] introduced the Iwahori-Hecke algebra as a first step in classifying the irreducible representations of finite Chevalley groups and reductive $p$-adic Lie groups. Subsequent work (e.g. [Cu1] [KL] [LV]) has established Hecke algebras as fundamental tools in the representation theory of Lie groups and Lie algebras, and advances on subfactors and quantum groups by Jones [Jo1], Jimbo [Ji], and Drinfeld [Dr] gave Hecke algebras a central role in knot theory [Jo2], statistical mechanics [Jo3], mathematical physics, and operator algebras. This paper considers a generalization of the classical Iwahori-Hecke algebra obtained by replacing the Borel subgroup $B$ with the unipotent subgroup $U$.

A Hecke algebra $\mathcal{H} = \mathcal{H}(G, U, M)$ is the centralizer algebra

$$\mathcal{H} = \text{End}_G(\text{Ind}_U^G(M)),$$

where $G$ is a finite group, $U$ is a subgroup of $G$ and $M$ is a simple $U$-module. This paper addresses the cases where

$$G = GL_n(\mathbb{F}_q), \quad U = \left\{ \begin{pmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \right\}, \quad \text{and} \quad \dim(M) = 1.$$

For each $M$ there exist a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ and a linear character $\psi_\mu : U \to \mathbb{C}^\ast$ such that

$$\text{End}_G(\text{Ind}_U^G(M)) \cong \text{End}_G(\text{Ind}_U^G(\psi_\mu)) = \mathcal{H}_\mu.$$

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In studying these algebras, this paper combines generalizations of Iwahori-Hecke algebra techniques with combinatorial tools related to the representation theory of the symmetric group (e.g. partitions, tableaux, and the RSK correspondence).

The main results of this paper are

(a) Let \( \{ T_v : v \in N_\mu \} \) be the standard double-coset basis for \( \mathcal{H}_\mu \). Then there is a bijection

\[
N_\mu \leftrightarrow M_\mu = \left\{ \begin{array}{l}
\ell \times \ell \text{ matrices with monic polynomial entries } a_{ij}(X) \in \mathbb{F}_q[X] \text{ such that } \\
a_{ij}(0) \neq 0 \text{ and both degree row sums } \\
\text{and degree column sums are equal to } \mu
\end{array} \right\}
\]

(See Section 3 for details).

(b) Let the set \( \mathcal{H}_\mu \) index the irreducible \( \mathcal{H}_\mu \)-modules \( \mathcal{H}_{\mu\lambda} \) and the set \( \mathcal{H}_{\mu\lambda} \) index a basis of the irreducible module \( \mathcal{H}_{\mu\lambda} \). There is a combinatorial bijection

\[
N_\mu \leftrightarrow \left\{ \begin{array}{l}
\text{Pairs } (P, Q) \text{ such that } \\
P, Q \in \mathcal{H}_{\mu\lambda}, \lambda \in \mathcal{H}_\mu
\end{array} \right\}
\]

that generalizes the classical RSK correspondence and gives a combinatorial realization of the representation theoretic identity

\[
\dim(\mathcal{H}_\mu) = \sum_{\lambda \in \mathcal{H}_\mu} \dim(\mathcal{H}_{\mu\lambda})^2.
\]

(c) The algebra \( \mathcal{H}_\mu \) has a large commutative subalgebra

\[
\mathcal{L}_\mu \cong \mathcal{H}_{(\mu_1)} \otimes \mathcal{H}_{(\mu_2)} \otimes \cdots \otimes \mathcal{H}_{(\mu_k)};
\]

whose presence suggests a weight space decomposition of \( \mathcal{H}_\mu \)-modules.

Section 2 reviews some basic results used in this paper, including Hecke algebras, partitions and the classical RSK correspondence. Section 3 defines \( \mathcal{H}_\mu \) and gives an explicit construction of the map \( N_\mu \leftrightarrow M_\mu \). After using Zelevinsky’s theorem [Ze, Theorem 12.1] to give a combinatorial description of the sets \( \mathcal{H}_\mu \) and \( \mathcal{H}_{\mu\lambda} \), Section 4 proves the bijection in (b). Section 5 gives a proof of Zelevinsky’s theorem. This paper concludes in Section 6 by providing the subalgebra \( \mathcal{L}_\mu \subseteq \mathcal{H}_\mu \) and describing a corresponding weight space decomposition of \( \mathcal{H}_\mu \)-modules.

Both the Yokonuma algebra \( \mathcal{H}_{(1^n)} \) [Yo2] and the Hecke algebra \( \mathcal{H}_{(n)} \) associated to the Gelfand-Graev representation of \( G \) [St] are examples of unipotent Hecke algebras. In [Yo2], Yokonuma gave a presentation of \( \mathcal{H}_{(1^n)} \) that generalized the usual presentation of the classical Iwahori-Hecke algebra, and recently Jujumaya [Ju] constructed an alternate set of generators and relations. However, a presentation for arbitrary \( \mathcal{H}_\mu \) is still unknown. Even the commutative algebra \( \mathcal{H}_{(n)} \) [St, Yo1] does not yet have a “nice” set of multiplication relations.

The representation theory of the Gelfand-Graev Hecke algebra is closely related to Green polynomials [Cu2] and, in the GL_2 case, to Kloosterman sums [CS]. On the other hand, the representation theory of the Yokonuma algebra generalizes that of the classical Iwahori-Hecke algebra. In what promises to be a combinatorially rich area of study, analyzing the combinatorial representation theory of the \( \mathcal{H}_\mu \) and their general type analogues should have an impact similar in scope to the applications of the classical Iwahori-Hecke algebra.
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2 Preliminaries: Hecke algebras, some combinatorics of the symmetric group, and $\text{GL}_n(\mathbb{F}_q)$

Hecke algebras. Let $U$ be a subgroup of a finite group $G$. If $M$ is an irreducible $U$-module, then the Hecke algebra $H = H(G,U,M)$ is

$$H = \text{End}_G(\text{Ind}_U^G(M)) \cong e\mathbb{C}Ge,$$

where $e$ is an idempotent of $\mathbb{C}U$ such that $M \cong \mathbb{C}Ue$ [CR, (3.19)]. If $N$ is a set of double coset representatives for the cosets $U \backslash G/U$, then the set

$$\{T_v = eve : v \in N, e \neq 0\}$$

is a basis for $H$ [CR, (11.30)].

Let $\mathcal{H}$ be an indexing set for the irreducible $H$-modules $H^\lambda$. As a $(G,\mathcal{H})$-bimodule,

$$\mathbb{C}Ge \cong \text{Ind}_U^G(M) \cong \bigoplus_{\lambda \in \mathcal{H}} G^\lambda \otimes H^\lambda,$$

where the $G^\lambda$ are the irreducible constituents of $\text{Ind}_U^G(M)$ [GW, Thm 3.3.7]; it follows that

$$\dim(H^\lambda) = \text{multiplicity of } G^\lambda \text{ in the } G\text{-module } \text{Ind}_U^G(M).$$

Compositions, partitions and tableaux. A composition $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ is a sequence of positive integers. The size of $\mu$ is $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_r$, the length of $\mu$ is $\ell(\mu) = r$ and

$$\mathcal{B}_\mu = \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_r\}.$$  

(2.3)

If $|\mu| = n$, then $\mu$ is a composition of $n$ and we write $\mu \models n$. View $\mu$ as a collection of boxes aligned to the left; for example, if

$$\mu = (2, 5, 3, 4) = \begin{array}{cccc}
\hline
\hline
\end{array},$$

then $|\mu| = 14$, $\ell(\mu) = 4$ and $\mathcal{B}_\mu = \{2, 7, 10, 14\}$. Alternatively, $\mathcal{B}_\mu$ coincides with the numbers in the boxes at the end of the rows in the diagram

$\begin{array}{cccc}
1 & 2 \\
3 & 4 & 5 & 7 \\
8 & 9 & 10 & 11 \\
12 & 13 & 14 \\
\hline
\end{array}$.
A partition \( \nu = (\nu_1, \nu_2, \ldots, \nu_r) \) is a composition where \( \nu_1 \geq \nu_2 \geq \cdots \geq \nu_r > 0 \). If \( |\nu| = n \), then \( \nu \) is a partition of \( n \) and we write \( \nu \vdash n \). Let

\[
P = \{ \text{partitions} \} \quad \text{and} \quad P_n = \{ \nu \vdash n \}.
\]

Suppose \( \nu \in \mathcal{P} \). The conjugate partition \( \nu' = (\nu'_1, \nu'_2, \ldots, \nu'_r) \) is given by

\[
\nu'_i = \text{Card}\{ j : \nu_j \geq i \}.
\]

In terms of diagrams, \( \nu' \) is the collection of boxes obtained by flipping \( \nu \) across its main diagonal. For example,

if \( \nu = \)

\[
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 6 & & \\
3 & 4 & 7 & \\
4 & 6 & & \\
5 & & & \\
\end{array}
\]

then \( \nu' = \)

\[
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 6 & & \\
3 & 4 & 7 & \\
4 & 6 & & \\
5 & & & \\
\end{array}
\]

A column strict tableau \( Q \) of shape \( \nu \) is a filling of the boxes of \( \nu \) by positive integers such that

(a) the entries strictly increase along columns,

(b) the entries weakly increase along rows.

The weight of \( Q \) is the composition \( \text{wt}(Q) = (\text{wt}(Q)_1, \text{wt}(Q)_2, \ldots) \) given by

\[
\text{wt}(Q)_i = \text{number of } i \text{ in } Q.
\]

For example,

\[
Q = \begin{array}{cccc}
1 & 1 & 1 & 2 & 4 & 4 \\
2 & 2 & 6 & & \\
3 & 4 & 7 & & \\
4 & 6 & & & \\
5 & & & & \\
\end{array}
\]

has \( \text{wt}(Q) = (3, 3, 1, 4, 1, 2, 1) \).

Suppose \( \nu, \mu \) are partitions. If \( \nu_i \geq \mu_i \) for all \( 1 \leq i \leq \ell(\mu) \), then the skew partition \( \nu/\mu \) is given by

\[
\nu/\mu = (\nu_1 - \mu_1, \nu_2 - \mu_2, \ldots, \nu_{\ell(\nu)} - \mu_{\ell(\nu)}),
\]

where \( \mu_k = 0 \) for all \( k \geq \ell(\mu) \). In terms of boxes, represent \( \nu/\mu \) by removing \( \mu \) from the upper left-hand corner of the diagram \( \nu \), so if

\[
\nu = \begin{array}{cccc}
1 & 1 & 1 & 2 & 4 & 4 \\
2 & 2 & 6 & & \\
3 & 4 & 7 & & \\
4 & 6 & & & \\
5 & & & & \\
\end{array}
\quad \text{and} \quad
\mu = \begin{array}{cccc}
1 & 1 & 1 & 2 & 4 & 4 \\
2 & 2 & 6 & & \\
3 & 4 & 7 & & \\
4 & 6 & & & \\
5 & & & & \\
\end{array}
\]

then \( \nu/\mu = \)

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 & 4 & 4 \\
2 & 6 & & \\
3 & 4 & 7 & \\
4 & 6 & & \\
5 & & & & \\
\end{array}
\]

A column strict tableaux of shape \( \nu/\mu \) is a filling of \( \nu/\mu \) satisfying (a) and (b) above.

**Symmetric Functions.** The symmetric group \( S_n \) acts on the infinite set of variables \( \{x_1, x_2, \ldots\} \) by permuting the indices \( \leq n \) and fixing those \( > n \). Let

\[
\Lambda_C(x) = \{ f \in \mathbb{C}[[x_1, x_2, \ldots]] : w(f) = f, \text{ permutations } w \}
\]
be the ring of symmetric functions in the variables \( \{x_1, x_2, \ldots \} \). Let
\[
e_r(x) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \quad \text{and} \quad p_s(x) = \sum_{1 \leq i} x_i^s, \quad r, s \in \mathbb{Z}_{\geq 0},
\]
be the \( r \)th elementary symmetric function and the \( s \)th power sum symmetric function, respectively. For a partition \( \nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P} \), let
\[
e_\nu(x) = e_{\nu_1}(x) e_{\nu_2}(x) \cdots e_{\nu_\ell}(x), \quad p_\nu(x) = p_{\nu_1}(x) p_{\nu_2}(x) \cdots p_{\nu_\ell}(x)
\]
and let
\[
s_\nu(x) = \det(e_{\nu'_i - i + j}(x)) \quad (2.5)
\]
be the Schur function corresponding to \( \nu \). The ring
\[
\Lambda_C(x) = \mathbb{C}\text{-span}\{e_\nu(x)\} = \mathbb{C}\text{-span}\{p_\nu(x)\} = \mathbb{C}\text{-span}\{s_\nu(x)\}, \quad (2.6)
\]
and Pieri’s rule says that if \( \nu \in \mathcal{P} \), then
\[
s_\nu(x)s_{(\nu)}(x) = \sum_{\text{sh}(P)=\gamma/\nu \atop \text{wt}(P)=(\nu)} s_\gamma(x). \quad [\text{Ma, I.5.16}] \quad (2.7)
\]
For each \( t \in \mathbb{C} \) and partition \( \nu \), let \( P_\nu(x; t) \) denote the Hall-Littlewood symmetric function [Ma, III.2]. Since a precise definition is not necessary for this paper, it suffices to remark that
\[
P_\nu(x; 0) = s_\nu(x), \quad P_{(1^n)}(x; t) = e_n(x)
\]
and for each \( t \in \mathbb{C} \)
\[
\Lambda_C(x) = \mathbb{C}\text{-span}\{P_\nu(x; t)\}.
\]
(For additional details, see [Ma, Chapter I] on symmetric functions and [Ma, Chapter III] on Hall-Littlewood functions).

**Remark.** It is usually sufficient to let \( \{x_1, x_2, \ldots, x_K\} \) be a finite variable set (for \( K \) much bigger than \( |G_n| \)); think of symmetric functions as polynomials rather than formal power series by setting \( x_j = 0 \) for \( j > K \) in the definitions above. While Theorem 5.2 requires the infinite definition, I urge the reader to think in terms of the finite version everywhere else.

**RSK correspondence.** The classical RSK correspondence provides a combinatorial proof of the identity
\[
\prod_{i,j>0} \frac{1}{1-x_i y_j} = \sum_{\nu \vdash n, n \geq 0} s_\nu(x)s_\nu(y) \quad [\text{Kn}]
\]
by constructing a bijection between the matrices \( b \in M_k(\mathbb{Z}_{\geq 0}) \) and the set of pairs \( (P(b), Q(b)) \) of column strict tableaux with the same shape. The bijection is as follows.

If \( P \) is a column strict tableau and \( j \in \mathbb{Z}_{\geq 0} \), let \( P \leftarrow j \) be the column strict tableau given by the following algorithm
\begin{enumerate}
  \item Insert \( j \) into the the first column of \( P_k \) by displacing the smallest number \( \geq j \). If all numbers are \( < j \), then place \( j \) at the bottom of the first column.
  \item Iterate this insertion by inserting the displaced entry into the next column.
  \item Stop when the insertion does not displace an entry.
\end{enumerate}
A two-line array \( \left( \begin{array}{cccc}
i_1 & i_2 & \cdots & i_n \\
j_1 & j_2 & \cdots & j_n
\end{array} \right) \) is a two-rowed array with \( i_1 \leq i_2 \leq \cdots \leq i_n \) and \( j_k \geq j_{k+1} \) if \( i_k = i_{k+1} \). If \( b \in M_\ell(\mathbb{Z}_{\geq 0}) \), then let \( \tilde{b} \) be the two-line array with \( b_{ij} \) pairs \( \left( \begin{array}{c} i \\
j \end{array} \right) \).

For \( b \in M_\ell(\mathbb{Z}_{\geq 0}) \), suppose
\[
\tilde{b} = \left( \begin{array}{cccc}
i_1 & i_2 & \cdots & i_n \\
j_1 & j_2 & \cdots & j_n
\end{array} \right).
\]

Then the pair \((P(b), Q(b))\) is the final pair in the sequence
\[
(\emptyset, \emptyset) = (P_0, Q_0), (P_1, Q_1), (P_2, Q_2), \ldots, (P_n, Q_n) = (P(b), Q(b)),
\]
where \((P_k, Q_k)\) is a pair of column strict tableaux with the same shape given by
\[
P_k = P_{k-1} \leftarrow j_k \quad \text{and} \quad Q_k \text{ is defined by } \text{sh}(Q_k) = \text{sh}(P_k) \text{ with } i_k \text{ in the new box } \text{sh}(Q_k) / \text{sh}(Q_{k-1}).
\]

For example,
\[
b = \left( \begin{array}{ccc}
1 & 1 & 0 \\
0 & 0 & 2 \\
0 & 1 & 0
\end{array} \right) \quad \text{corresponds to} \quad \tilde{b} = \left( \begin{array}{cccc}
1 & 1 & 2 & 2 & 3 \\
2 & 1 & 3 & 3 & 2
\end{array} \right)
\]
and provides the sequence
\[
(\emptyset, \emptyset), (1_{2,1}, 1_{2,1}), \left( \begin{array}{cc}
1 & 2 \\
3 & 1
\end{array} \right), \left( \begin{array}{cc}
1 & 2 \\
3 & 3
\end{array} \right), \left( \begin{array}{cc}
1 & 2 \\
3 & 2
\end{array} \right), \left( \begin{array}{cc}
1 & 2 \\
3 & 1
\end{array} \right), \left( \begin{array}{cc}
1 & 3 \\
2 & 2
\end{array} \right)
\]
so that
\[
(P(b), Q(b)) = \left( \begin{array}{cc}
1 & 2 \\
2 & 3
\end{array} \right), \left( \begin{array}{cc}
1 & 1 \\
2 & 2
\end{array} \right).
\]

The general linear group. Let \( G = \text{GL}_n(\mathbb{F}_q) \), where \( \mathbb{F}_q \) is the finite field with \( q \) elements. Let
\[
U = \left\{ \left( \begin{array}{cccc}
1 & * & \cdots & * \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & * \\
0 & \cdots & 0 & 1
\end{array} \right) \right\} \subset G
\]
be the subgroup of unipotent, upper-triangular matrices. For \( 1 \leq i \neq j \leq n \), let \( x_{ij}(t) \) be the matrix with ones on the diagonal, \( t \) in the \((i,j)\)th position and zeroes elsewhere. Then
\[
U = \langle x_{ij}(t) : 1 \leq i < j \leq n \rangle.
\]

The group \( G \) has a double-coset decomposition given by
\[
G = \bigsqcup_{w \in N} U w U, \quad \text{where} \quad N = \left\{ \begin{array}{c}
n \times n \text{ matrices with entries from } \mathbb{F}_q \text{ and exactly one nonzero entry in each row and column}
\end{array} \right\}.
\]

If \( T \subseteq N \) is the subgroup of diagonal matrices and \( W \subseteq N \) is the the subgroup of permutation matrices, then \( N = WT \) and \( TU = N_G(U) \) is the normalizer of \( U \) in \( G \). If necessary, specify the size of the group by a subscript such as \( G_n, U_n, W_n, \) etc. Let \( w(k) \in W_k \) be the \( k \times k \) matrix
\[
(w(k))_{ij} = \delta_{j,n-i+1}.
\]

For example, \( w(3) = \left( \begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array} \right) \) (2.9)
For $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \vdash n$,
\[
P_\mu = \left\{ \begin{pmatrix} g_1 & \ast \\ g_2 & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & g_\ell \end{pmatrix} : g_i \in G_{\mu_i} = \text{GL}_{\mu_i}(\mathbb{F}_q) \right\} \tag{2.10}
\]
is a parabolic subgroup of $G$. The Levi subgroup and the unipotent radical of $P_\mu$ are
\[
L_\mu = \left\{ \begin{pmatrix} g_1 & 0 & \cdots \\ g_2 & \ddots & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & g_\ell \end{pmatrix} : g_i \in G_{\mu_i} \right\} \quad \text{and} \quad U_\mu = \left\{ \begin{pmatrix} \text{Id}_{\mu_1} & & * \\ & \ddots & \\ & \ddots & \\ 0 & & \text{Id}_{\mu_\ell} \end{pmatrix} \right\}, \tag{2.11}
\]
respectively, where $\text{Id}_k$ is the $k \times k$ identity matrix. Note that $P_\mu = L_\mu U_\mu$ and $P_\mu = \mathbf{N}_G(U_\mu)$.

3 An indexing for the standard basis of $\mathcal{H}_\mu$

Let $G = \text{GL}_n(\mathbb{F}_q)$. Fix a nontrivial character $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$ of the additive group of $\mathbb{F}_q$. Let $\mu \vdash n$ and $\mathcal{B}_\mu$ be as in (2.3). Since $x_{ij}(t) \in [U, U]$ for all $j > i + 1$, the map $\psi_\mu : U \to \mathbb{C}^*$, defined by
\[
\psi_\mu(x_{ij}(t)) = \begin{cases} 
\psi(t), & \text{if } j = i + 1, i \notin \mathcal{B}_\mu, \\
1, & \text{otherwise},
\end{cases} \tag{3.1}
\]
is a linear character of $U$. Let
\[
\mathcal{H}_\mu = \text{End}_G(\text{Ind}_U^G(\psi_\mu)) \cong e_\mu \mathbb{C} \mathcal{C} e_\mu, \quad \text{where } e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1})u. \tag{3.2}
\]

The classical examples of unipotent Hecke algebras are the Yokonuma algebra $\mathcal{H}_{(1^n)}$ [Yo2] and the Gelfand-Graev Hecke algebra $\mathcal{H}_{(n)}$ [St]. A fundamental result is

**Theorem 3.1 ([GG],[Yo1],[St]).** For all $n > 0$, $\mathcal{H}_{(n)}$ is commutative.

This theorem will follow from Theorem 5.3.

An analysis of (2.1) implies that $\{T_v = e_\mu v e_\mu : v \in N_\mu\}$ is a basis for $\mathcal{H}_\mu$, where $N_\mu = \{v \in N : u, uvv^{-1} \in U \text{ implies } \psi_\mu(u) = \psi_\mu(vvv^{-1})\}$. [CR, (11.30)]

\[
\text{Suppose } a \in M_\ell(\mathbb{F}_q[X]) \text{ is an } \ell \times \ell \text{ matrix with polynomial entries. Let } d(a_{ij}) \text{ be the degree of the polynomial } a_{ij}. \text{ Define the degree row sums and the degree column sums of } a \text{ to be the compositions}
\]
\[
d^{-}(a) = (d^{-}(a)_{1}, d^{-}(a)_{2}, \ldots, d^{-}(a)_{\ell}) \quad \text{and} \quad d^{+}(a) = (d^{+}(a)_{1}, d^{+}(a)_{2}, \ldots, d^{+}(a)_{\ell}),
\]
where
\[
d^{-}(a)_{i} = \sum_{j=1}^{\ell} d(a_{ij}) \quad \text{and} \quad d^{+}(a)_{j} = \sum_{i=1}^{\ell} d(a_{ij}).
\]

Let
\[
M_\mu = \{a \in M_\ell(\mathbb{F}_q[X]) : d^{-}(a) = d^{+}(a) = \mu, a_{ij} \text{ monic, } a_{ij}(0) \neq 0\}. \tag{3.4}
\]
For example,
\[
\begin{pmatrix}
X+1 & 1 & 1 & X+2 \\
X+3 & X^3+2X+3 & 1 & X+2 \\
1 & X^2+4X+2 & X+2 & 1 \\
1 & 1 & X^2+3X+1 & X^2+2
\end{pmatrix}
\]

is as in (3.5). Let
\[v\) = w_{(n)}(a_0w_{(i_1)} \oplus a_1w_{(i_2-i_1)} \oplus \cdots \oplus a_rw_{(n-i_r)}) \in N,
\]
where \(w_{(k)} \in W_k\) is as in (2.9) and by convention \(v_{(1)}\) is the matrix with height and width 0.
For example,
\[
v_{(a+bX^3+cX^4+x^6)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & a & 0 & 0 \\
0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & c & 0 \\
0 & 0 & 0 & 0 & c \\
0 & 0 & b & 0 & 0 \\
0 & a & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0
\end{pmatrix}
\]

For each \(a \in M_\mu\) construct a matrix \(v_a \in N\) by partitioning the rows and columns of an \(n \times n\) matrix according to \(\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \models n\) and setting
\[
v_a = \begin{pmatrix}
v_{(a_1)} & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & v_{(a_\mu)} & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},
\]
where \(a_{ij}\) is the \((i, j)\)th entry of \(a\) and \(v_{(a_{ij})}\) is as in (3.5).

**Theorem 3.2.** The map
\[
M_\mu \rightarrow N_\mu \\
a \mapsto v_a,
\]
given by (3.6) is a bijection.
Remark. When $\mu = (n)$ this theorem says that the map $(f) \mapsto v(f)$ of (3.5) is a bijection between $M_{(n)}$ and $N_{(n)}$.

Proof. Using the remark following the theorem, it is straightforward to reconstruct $a$ from $v_a$. Therefore the map is invertible, and it suffices to show

(a) the map is well-defined ($v_a \in N_{\mu}$),

(b) the map is surjective.

To show (a) and (b), we investigate the matrices $N_{\mu}$. Suppose $v \in N_{\mu}$. Let

\[ v_i = \text{the nonzero entry in the } i\text{th column of } v, \]
\[ v(i) = \text{the row number of the nonzero entry in the } i\text{th column of } v, \]

so that $\{v_1, v_2, \ldots, v_n\}$ are the nonzero entries of $v$ and $(v(1), v(2), \ldots, v(n))$ is the permutation determined by setting all the nonzero entries of $v$ to 1. By (3.1),

\[
\psi_\mu(x_{ij}(t)) = \begin{cases} 
\psi(t), & \text{if } j = i + 1 \text{ and } i \notin B_\mu, \\
1, & \text{otherwise.}
\end{cases} \tag{A}
\]

Recall that $v \in N_{\mu}$ if and only if $u, u_{v^{-1}} \in U$ implies $\psi_\mu(u) = \psi_\mu(v u v^{-1})$. That is, $v \in N_{\mu}$ if and only if for all $1 \leq i < j \leq n$ such that $v(i) < v(j)$,

\[
\psi_\mu(x_{ij}(t)) = \psi_\mu(x_{ij}(t) v^{-1}) \\
= \psi_\mu(x_{v_i v_j}(v_i v_j^{-1})) \\
= \begin{cases} 
\psi(v_i v_j^{-1}), & \text{if } v(j) = v(i) + 1 \text{ and } v(i) \notin B_\mu, \\
1, & \text{otherwise.}
\end{cases} \tag{B}
\]

Compare (A) and (B) to obtain that $v \in N_{\mu}$ if and only if for all $1 \leq i < j \leq n$ such that $v(i) < v(j)$,

(i) If $i \notin B_\mu$ and $v(i) \in B_\mu$, then $j \neq i + 1$,

(ii) If $i \in B_\mu$ and $v(i) \notin B_\mu$, then $v(j) \neq v(i) + 1$,

(iii) If $i, v(i) \notin B_\mu$, then $j = i + 1$ if and only if $v(j) = v(i) + 1$,

(iii)' If $i, v(i) \notin B_\mu$ and $v(j) = v(i) + 1$, then $v_i = v_{i+1}$.

We can visualize the implications of the conditions (i)–(iii)' in the following way. Partition the rows and columns of $v \in N_{\mu}$ by $\mu$. For example, $\mu = (2, 3, 1)$ partitions $v$ according to

\[
\begin{pmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{pmatrix}.
\]
Suppose the nonzero entry of \( v \) in column \( i \) is above a horizontal line but not next to a vertical line. Then condition (i) implies that \( v(i + 1) < v(i) \), so

\[
\begin{array}{cccc}
0 & * & 0 & * \\
0 & * & 0 & *\\
0 & * & 0 & 0 \\
\end{array}
\]

where \( a \) is the nonzero entry and * indicates possible locations for nonzero entries in the next column.

(I)

Similarly, condition (ii) implies

\[
\begin{array}{cccc}
0 & 0 & 0 & a \\
0 & 0 & 0 & *
\end{array}
\]

and conditions (iii) and (iii)' imply

\[
\begin{array}{cccc}
0 & * & 0 & 0 \\
0 & * & 0 & 0 \\
0 & * & 0 & 0 \\
0 & 0 & 0 & a \\
0 & 0 & 0 & a \\
0 & 0 & 0 & 0 \\
* & * & 0 & 0 \\
\end{array}
\]

(III)

In the case \( \mu = (n) \) condition (III) implies that every \( v \in N(n) \) is of the form

\[
\begin{pmatrix}
0 & a_1 & \cdots & a_r \\
a_1 & 0 & \cdots & 0 \\
a_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix} = w(n)(a_1w_{(i_1)} \oplus a_2w_{(i_2)} \oplus \cdots \oplus a_rw_{(i_r)}),
\]

where \( (i_1, i_2, \ldots, i_r) = n \), \( a_i \in \mathbb{F}_q^* \), and \( w_{(k)} \in W_k \) is as in (2.9). In fact, this observation proves that the map \( (f) \mapsto v(f) \) is a bijection between \( M(n) \) and \( N(n) \) (mentioned in the remark).

Note that since the matrices \( v_a \) satisfy (I), (II) and (III), \( v_a \in N_\mu \), proving (a). On the other hand, (I), (II) and (III) imply that each \( v \in N_\mu \) must be of the form \( v = v_a \) for some \( a \in M_\mu \), proving (b).

4 An RSK-insertion via the representation theory of \( \mathcal{H}_\mu \)

Let \( S \) be a set. An \( S \)-partition \( \lambda = (\lambda^{(s_1)}, \lambda^{(s_2)}, \ldots) \) is a sequence of partitions indexed by the elements of \( S \). Let

\[ \mathcal{P}^S = \{ \text{\( S \)-partitions} \}. \quad (4.1) \]

The following discussion defines two sets \( \Theta \) and \( \Phi \), so that \( \Theta \)-partitions index the irreducible characters of \( G \) and \( \Phi \)-partitions index the conjugacy classes of \( G \).

Let \( L_n = \text{Hom}(\mathbb{F}_q^n, \mathbb{C}^*) \) be the character group of \( \mathbb{F}_q^n \). If \( \gamma \in L_m \), then let

\[
\gamma(r) : \mathbb{F}_q^{mr} \rightarrow \mathbb{C}^* \\
x \mapsto \gamma(x^{1+q^r+q^{2r}+\cdots+q^{m(r-1)}})
\]
Thus if \( n = mr \), then we may view \( L_m \subseteq L_n \) by identifying \( \gamma \in L_m \) with \( \gamma(r) \in L_n \). Define
\[
L = \bigcup_{n \geq 0} L_n.
\]

The Frobenius maps are
\[
F : \overline{\mathbb{F}}_q \rightarrow \overline{\mathbb{F}}_q \quad \text{and} \quad F : L \rightarrow L
\]
where \( \overline{\mathbb{F}}_q \) is the algebraic closure of \( \mathbb{F}_q \).

The map
\[
\Phi = \left\{ f \in \mathbb{C}[t] : f \text{ is monic, irreducible and } f(0) \neq 0 \right\} \quad \text{and} \quad \Theta = \{ \text{F-orbits in } L \}. \tag{4.2}
\]

If \( \eta \) is a \( \Phi \)-partition and \( \lambda \) is a \( \Theta \)-partition, then let
\[
|\eta| = \sum_{f \in \Phi} d(f)|\eta(f)| \quad \text{and} \quad |\lambda| = \sum_{\varphi \in \Theta} |\varphi||\lambda(\varphi)|
\]
be the size of \( \eta \) and \( \lambda \), respectively. Let the sets \( \mathcal{P}^\Phi \) and \( \mathcal{P}^\Theta \) be as in (4.1) and let
\[
\mathcal{P}_n^\Phi = \{ \eta \in \mathcal{P}^\Phi : |\eta| = n \} \quad \text{and} \quad \mathcal{P}_n^\Theta = \{ \lambda \in \mathcal{P}^\Theta : |\lambda| = n \}. \tag{4.3}
\]

**Theorem 4.1 (Green [Gr]).** Let \( G_n = \text{GL}_n(\mathbb{F}_q) \).

(a) \( \mathcal{P}_n^\Phi \) indexes the conjugacy classes \( K^\Phi \) of \( G_n \),

(b) \( \mathcal{P}_n^\Theta \) indexes the irreducible \( G_n \)-modules \( G_n^\lambda \).

Suppose \( \lambda \in \mathcal{P}^\Theta \). A column strict tableau \( P = (P^{(\varphi_1)}, P^{(\varphi_2)}, \ldots) \) of shape \( \lambda \) is a column strict filling of \( \lambda \) by positive integers. That is, \( P^{(\varphi)} \) is a column strict tableau of shape \( \lambda^{(\varphi)} \). Write \( \text{sh}(P) = \lambda \). The weight of \( P \) is the composition \( \text{wt}(P) = (\text{wt}(P)_1, \text{wt}(P)_2, \ldots) \) given by
\[
\text{wt}(P)_i = \sum_{\varphi \in \Theta} |\varphi| \left( \begin{array}{c} \text{number of} \ i \ \text{in} \ P^{(\varphi)} \end{array} \right).
\]

If \( \lambda \in \mathcal{P}^\Theta \) and \( \mu \) is a composition, then let
\[
\mathcal{H}^\lambda_{\mu} = \{ \text{column strict tableaux } P : \text{sh}(P) = \lambda, \text{wt}(P) = \mu \} \tag{4.4}
\]
and
\[
\mathcal{H}_{\mu} = \{ \lambda \in \mathcal{P}^\Theta : \mathcal{H}^\lambda_{\mu} \text{ is not empty} \}. \tag{4.5}
\]

The following theorem is a consequence of (2.2) and a theorem proved by Zelevinsky [Ze] (see Theorem 5.1). A proof of Zelevinsky’s theorem is in Section 5.
Theorem 4.2. The set $\tilde{H}_\mu$ indexes the irreducible $H_\mu$-modules $H_\mu^\lambda$ and
\[ \dim(H_\mu^\lambda) = |\tilde{H}_\mu^\lambda|. \]

The $(H_\mu, H_\mu)$-bimodule decomposition
\[ H_\mu \cong \bigoplus_{\lambda \in \tilde{H}_\mu} H_\mu^\lambda \otimes H_\mu^\lambda \]
implies $|N_\mu| = \dim(H_\mu) = \bigoplus_{\lambda \in \tilde{H}_\mu} \dim(H_\mu^\lambda)^2 = \sum_{\lambda \in \tilde{H}_\mu} |\tilde{H}_\mu^\lambda|^2$.

Theorem 4.3, below, gives a combinatorial proof of this identity.

Encode each matrix $a \in M_\mu$ as a $\Phi$-sequence
\[(a^{(f_1)}, a^{(f_2)}, \ldots), \quad f_i \in \Phi,\]
where $a^{(f)} \in M_{(\mu)}(\mathbb{Z}_{\geq 0})$ is given by
\[a^{(f)}_{ij} = \text{highest power of } f \text{ dividing } a_{ij}.\]

Note that this is an entry by entry “factorization” of $a$ such that
\[a_{ij} = \prod_{f \in \Phi} f^{a^{(f)}_{ij}}.\]

Recall from page 5 the classical RSK correspondence
\[ M_{(\mathbb{Z}_{\geq 0})} \rightarrow \{ \text{Pairs } (P, Q) \text{ of column strict} \}
\quad \{ \text{tableaux of the same shape} \} \]
\[ b \mapsto (P(b), Q(b)). \]

Theorem 4.3. For $a \in M_\mu$, let $P(a)$ and $Q(a)$ be the $\Phi$-column strict tableaux given by
\[ P(a) = (P(a^{(f_1)}), P(a^{(f_2)}), \ldots) \quad \text{and} \quad Q(a) = (Q(a^{(f_1)}), Q(a^{(f_2)}), \ldots) \quad \text{for } f_i \in \Phi. \]

Then the map
\[ N_\mu \rightarrow M_\mu \rightarrow \{ \text{Pairs } (P, Q) \text{ of } \Phi\text{-column} \}
\quad \{ \text{strict tableaux of the same} \}
\quad \{ \text{shape and weight } \mu \} \]
\[ v \mapsto a_v \mapsto (P(a_v), Q(a_v)), \]
is a bijection, where the first map is the inverse of the bijection of Theorem 3.2.

By the construction above, the map is well-defined and since all the steps are invertible, the map is a bijection.

For example, suppose $\mu = (7, 5, 3, 2)$ and $f, g, h \in \Phi$ are such that $d(f) = 1$, $d(g) = 2$, and $d(h) = 3$. Then
\[ a_v = \begin{pmatrix} g & f^2 h & 1 & 1 \\ h & 1 & g & 1 \\ 1 & 1 & f & f^2 \\ g & 1 & 1 & 1 \end{pmatrix} \in M_{(\mu)} \]
corresponds to the sequence

$$(a_v^{(f_1)}, a_v^{(f_2)}, \ldots) = \left( \begin{array}{cccc} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right)^{(f)}, \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right)^{(g)}, \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)^{(h)}$$

and

$$(P(a_v), Q(a_v)) = \left( \begin{array}{cccc} 2 & 2 & 3 \\ 2 & 2 & 2 \end{array} \right)^{(f)}, \left( \begin{array}{cccc} 1 & 1 & 3 \\ 3 & 3 & 3 \end{array} \right)^{(g)}, \left( \begin{array}{cccc} 1 & 1 & 2 \\ 4 & 2 & 2 \end{array} \right)^{(g)}, \left( \begin{array}{cccc} 1 & 1 & 2 \\ 1 & 2 & 2 \end{array} \right)^{(h)}.$$  

5 Zelevinsky’s decomposition of $\text{Ind}^G_U(\psi_\mu)$

This section proves the theorem

**Theorem 5.1 (Zelevinsky [Ze]).** Let $U$ be the subgroup of unipotent upper-triangular matrices of $G = \text{GL}_n(F_q)$, $\mu \mid n$ and $\psi_\mu$ be as in (3.1).

$$\text{Ind}^G_U(\psi_\mu) = \bigoplus_{\lambda \in \mathcal{H}_\mu} (G^\lambda)^{\#[\mathcal{H}_\mu]}.$$  

Theorem 4.2 follows from this theorem and double-centralizer theory (2.2). The following will

(i) establish the necessary connection between symmetric functions and the representation theory of $G$,

(ii) prove Theorem 5.1 for the case when $\ell(\mu) = 1$,

(iii) generalize to arbitrary $\mu$.

**Preliminaries to the proof.** Suppose $\lambda \in \mathcal{P}^\Theta$ and $\eta \in \mathcal{P}^\Phi$ (see (4.3)). Let $\chi^\lambda$ be the irreducible character corresponding to the irreducible $G$-module $G^\lambda$ and let $\kappa^\eta$ be the characteristic function corresponding to the conjugacy class $K^\eta$ (see Theorem 4.1), given by

$$\kappa^\eta(g) = \begin{cases} 1, & \text{if } g \in K^\eta, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } g \in G_{[\eta]}.$$  

Define

$$R = \mathbb{C}\text{-span}\{\chi^\lambda : \lambda \in \mathcal{P}^\Theta\} = \mathbb{C}\text{-span}\{\kappa^\eta : \eta \in \mathcal{P}^\Phi\}.$$  

The space $R$ has an inner product defined by

$$\langle \chi^\lambda, \chi^\nu \rangle = \delta_{\lambda\nu},$$  

and multiplication

$$\chi^\lambda \circ \chi^\nu = \text{Ind}^G_{L^{(r,s)}}(\chi^\lambda \otimes \chi^\nu) = \text{Ind}^G_{L^{(r,s)}}\left( \text{Inf}_{L^{(r,s)}}(\chi^\lambda \otimes \chi^\nu) \right),$$  

for $\lambda \in \mathcal{P}^\Theta, \nu \in \mathcal{P}^\Theta_g$, (5.1)

where if $p \in P_\mu = L_\mu U_\mu$ decomposes as $p = lu$ for $u \in U_\mu, l \in L_\mu$, then

$$\text{Inf}^{P_\mu}_{L_\mu}(\chi^{\gamma_1} \otimes \cdots \otimes \chi^{\gamma_l})(p) = \chi^{\gamma_1}(l_1) \cdots \chi^{\gamma_l}(l_l),$$  

for $l = (l_1 \oplus \cdots \oplus l_l), l_i \in G_{\mu_i}, \gamma_i \in \mathcal{P}^\Theta_{\mu_i}$.  

13
For each \( \varphi \in \Theta \), let \( \{Y_{1}^{(\varphi)}, Y_{2}^{(\varphi)}, \ldots\} \) be an infinite set of variables, and let

\[
\Lambda \mathcal{C} = \bigotimes_{\varphi \in \Theta} \Lambda \mathcal{C}(Y^{(\varphi)}),
\]

where \( \Lambda \mathcal{C}(Y^{(\varphi)}) \) is the ring of symmetric functions in \( \{Y_{1}^{(\varphi)}, Y_{2}^{(\varphi)}, \ldots\} \) (see page 4). For each \( f \in \Phi \), define an additional set of variables \( \{X_{1}^{(f)}, X_{2}^{(f)}, \ldots\} \) such that the symmetric functions in the \( Y \) variables are related to the symmetric functions in the \( X \) variables by the transform

\[
p_{k}(Y^{(\varphi)}) = (-1)^{k|\varphi|-1} \sum_{x \in \mathbb{P}, k|\varphi|} \xi(x)p_{k|\varphi| \frac{\mu_{x}}{\mu_{r(x)}}}(X^{(f)}), \tag{5.2}
\]

where \( \xi \in \varphi, f_{x} \in \Phi \) is the irreducible polynomial that has \( x \) as a root, and \( p_{k}(X^{(f)}) = 0 \) if \( \frac{a}{b} \notin \mathbb{Z}_{\geq 0} \). Then

\[
\Lambda \mathcal{C} = \bigotimes_{f \in \Phi} \Lambda \mathcal{C}(X^{(f)}).
\]

For \( \nu \in \mathcal{P} \), let \( s_{\nu}(Y^{(\varphi)}) \) be the Schur function and \( P_{\nu}(X^{(f)}; t) \) be the Hall-Littlewood symmetric function (see page 4). Define

\[
s_{\lambda} = \prod_{\nu \in \Theta} s_{\lambda(\nu)}(Y^{(\varphi)}) \quad \text{and} \quad P_{\eta} = q^{-n(\eta)} \prod_{f \in \Phi} P_{\eta_{(f)}}(X^{(f)}; q^{-d(f)}), \tag{5.3}
\]

where \( n(\eta) = \sum_{f \in \Phi} d(f)n(\eta_{(f)}) \) and \( n(\mu) = \sum_{i=1}^{f}(i-1)\mu_{i} \), for \( \mu \) a composition. The ring

\[
\Lambda \mathcal{C} = \mathbb{C}\text{-span}\{s_{\lambda} : \lambda \in \mathcal{P}^{\Theta}\} = \mathbb{C}\text{-span}\{P_{\eta} : \eta \in \mathcal{P}^{\Phi}\}
\]

has an inner product given by

\[
\langle s_{\lambda}, s_{\nu} \rangle = \delta_{\lambda\nu}.
\]

**Theorem 5.2 (Green [Gr], Macdonald [Ma]).** The linear map

\[
\text{ch} : R \rightarrow \Lambda \mathcal{C}
\]

\[
\chi_{\lambda} \mapsto s_{\lambda}, \quad \text{for } \lambda \in \mathcal{P}^{\Theta}
\]

\[
\kappa_{\eta} \mapsto P_{\eta}, \quad \text{for } \eta \in \mathcal{P}^{\Phi}
\]

is an algebra isomorphism that preserves the inner product.

The unipotent conjugacy classes \( K^{\eta} \) satisfy \( \eta_{(f)} = \emptyset \) unless \( f = t - 1 \). Let

\[
\mathcal{U} = \mathbb{C}\text{-span}\{\kappa_{\eta} : \eta_{(f)} = \emptyset, \text{ unless } f = t - 1\} \subseteq R
\]

be the subalgebra of unipotent class functions. Note that by (5.3) and Theorem 5.2

\[
\text{ch}(\mathcal{U}) = \Lambda \mathcal{C}(X^{(t-1)}).
\]

Consider the projection \( \pi : R \rightarrow \mathcal{U} \) which is an algebra homomorphism given by

\[
(\pi \chi_{\lambda})(g) = \begin{cases} 
\chi_{\lambda}(g), & \text{if } g \in G \text{ is unipotent}, \\
0, & \text{otherwise},
\end{cases} \quad \lambda \in \mathcal{P}^{\Theta}.
\]
Then \( \tilde{\pi} = \pi \circ \text{ch}^{-1} : \Lambda_{\mathbb{C}} \to \mathbb{C} \) is given by

\[
\tilde{\pi}(p_k(Y^{(\varphi)})) = \tilde{\pi}\left(-1)^{k|\varphi|-1} \sum_{x \in \mathbb{F}_q^{k|\varphi|}} \xi(x)p_{k|\varphi|}(X^{(f_x)})\right) \quad (\text{by } (5.2))
\]

\[
= (-1)^{k|\varphi|-1}\xi(1)p_{k|\varphi|}(X^{(t-1)}) + 0
\]

\[
= (-1)^{k|\varphi|-1}p_{k|\varphi|}(X^{(t-1)}).
\]

### The decomposition of \( \text{Ind}_{U}^{G}(\psi_{(n)}) \).

The representation \( \text{Ind}_{U}^{G}(\psi_{(n)}) \) is the Gelfand-Graev module, and with (2.2), Theorem 5.3 proves that \( \mathcal{H}_{(n)} \) is commutative.

For \( \lambda \in \mathcal{P}^{\Theta} \), let

\[
\operatorname{ht}(\lambda) = \max\{\ell(\lambda^{(\varphi)}) : \varphi \in \Theta\}.
\]

#### Theorem 5.3.

\[
\text{ch}(\text{Ind}_{U}^{G}(\psi_{(n)})) = \sum_{\lambda \in \mathcal{P}^{\Theta}_{\mathbb{C}} \operatorname{ht}(\lambda) = 1} s_{\lambda}.
\]

**Proof.** Let

\[
\Psi : R \rightarrow \mathbb{C}, \quad \chi^\lambda \mapsto \langle \chi^\lambda, \text{Ind}_{U}^{G}(\psi_{(n)}) \rangle \quad \text{and} \quad \tilde{\Psi} : \Lambda_{\mathbb{C}} \xrightarrow{\text{ch}^{-1}} R \xrightarrow{\Psi} \mathbb{C}.
\]

For any group \( H \), let \( 1_{H} \) be the trivial character of \( H \), \( e_{H} = \frac{1}{|H|} \sum_{h \in H} h \), and \( \langle \chi, \gamma \rangle_{H} = \frac{1}{|H|} \sum_{h \in H} \chi(h) \gamma(h^{-1}) \), for all class functions \( \gamma \) and \( \chi \) of \( H \).

The proof is in six steps.

(a) \( \tilde{\Psi}(e_{k}(Y^{(1)})) = \delta_{k1} \), where 1 is the trivial character of \( \mathbb{F}_{q}^{*} \).

(b) \( \Psi(\chi^{\lambda}) = \dim(e_{(\alpha)}G^{\lambda}) \) for \( \lambda \in \mathcal{P}^{\Theta} \).

(c) \( \tilde{\Psi}(fg) = \tilde{\Psi}(f)\tilde{\Psi}(g) \) for all \( f, g \in \Lambda_{\mathbb{C}}(Y^{(1)}) \), where 1 is the trivial character of \( \mathbb{F}_{q}^{*} \).

(d) \( \Psi \circ \pi = \Psi \).

(e) \( \tilde{\Psi}(f(Y^{(\varphi)})) = \tilde{\Psi}(f(Y^{(1)})) \) for all \( f \in \Lambda_{\mathbb{C}}(Y^{(\varphi)}) \),

(f) \( \tilde{\Psi}(s_{\lambda}) = \delta_{\operatorname{ht}(\lambda)1} \).

(a) An argument similar to the argument in [Ma, pgs. 285-286] shows that

\[
\text{ch}^{-1}(e_{k}(Y^{(1)})) = 1_{G_{k}}
\]

(see [HR, Theorem 4.9 (a)] for details). Therefore, by Frobenius reciprocity and the orthogonality of characters,

\[
\tilde{\Psi}(e_{k}(Y^{(1)})) = \langle 1_{G_{k}}, \text{Ind}_{U}^{G}(\psi_{(n)}) \rangle = \langle 1_{U_{k}}, \psi_{(n)} \rangle_{U_{k}} = \delta_{k1}.
\]

(b) Since there exists an idempotent \( e \) such that \( G^{\lambda} \cong \mathbb{C}Ge \) and \( \text{Ind}_{U}^{G}(\psi_{(n)}) \cong \mathbb{C}Ge_{(n)} \), the map

\[
e_{(n)}\mathbb{C}Ge \rightarrow \text{Hom}_{\mathbb{C}}(G^{\lambda}, \mathbb{C}Ge_{(n)})
\]

\[
e_{(n)}ge \rightarrow \gamma_{g} : \mathbb{C}Ge \rightarrow \mathbb{C}Ge_{(n)}
\]

\[
x \mapsto x \gamma_{g}(x)
\]
is a vector space isomorphism (using an argument similar to the proof of [CR, (3.18)]). Thus,

\[ \Psi(\chi^s) = \langle \chi^s, \text{Ind}^G_U(\psi_{(n)}) \rangle = \dim(\text{Hom}_G(G^s, \text{Ind}^G_U(\psi_{(n)}))) = \dim(e_nCGe) = \dim(e_nG^s). \]

(c) By (a), \( \Psi(e_r(Y^{(1)})) \Psi(e_s(Y^{(1)})) = \delta_{r1}\delta_{s1} \). Since \( \Lambda_C(Y^{(1)}) = \mathbb{C}[e_1(Y^{(1)}), e_2(Y^{(1)}), \ldots] \), it therefore suffices to show that

\[ \Psi(e_r(Y^{(1)})e_s(Y^{(1)})) = \delta_{r1}\delta_{s1}. \]

Suppose \( r + s = n \) and let \( P = P_{(r,s)} \). Then

\[ \Psi(e_r(Y^{(1)})e_s(Y^{(1)})) = \Psi(\text{Ind}^G_{P_n}(1_P)) = \dim(e_nCGe_P). \]

Since \( T \subseteq P, e_P = e_{(1^n)}e_P, G = \bigsqcup_{w \in N} UvU, \) and \( N = WT \),

\[ e_nCGe_P = e_nCGe_{(1^n)}e_P = \mathbb{C}\text{-span}\{e_nwe_{(1^n)}e_P : w \in W\}. \]

If there exists \( 1 \leq i \leq n \) such that \( w(i) = w(i) + 1 \), then

\[ e_nwe_{(1^n)} = e_nwx_{i,i+1}(t)e_{(1^n)} = e_nx_{w(i),w(i)+1}(t)we_{(1^n)} = \psi(t)e_nwe_{(1^n)}. \]

Therefore, \( e_nwe_{(1^n)} = 0 \) unless \( w = w_n \). If \( r > 1 \) or \( s > 1 \), then there exists \( 1 \leq i \leq n \) such that \( x_{i,i+1}(t) \in P_{(r,s)}, \) so

\[ e_nw_ne_P = e_nw_ne_{(1^n)}e_P = e_nx_{n-i,n+i+1}(t)w_ne_P = \psi(t)e_nw_ne_P = 0. \]

In particular,

\[ \dim(e_nCGe_P) = 0. \]

If \( r = s = 1 \), then \( P_{(1,1)} \) is upper-triangular, so

\[ e_{(2)}w_{(2)}e_P \neq 0 \]

and \( \dim(e_{(2)}CGe_P) = 1 \), giving \( \Psi(e_r(Y^{(1)})e_s(Y^{(1)})) = \delta_{r1}\delta_{s1} \).

(d) By Frobenius reciprocity,

\[ \langle \chi^s, \text{Ind}^G_{U_n}(\psi_{(n)}) \rangle = \langle \text{Res}^G_U(\pi(\chi^s)), \psi_{(n)} \rangle_{U_n} = \langle \text{Res}^G_U(\pi(\chi^s)), \psi_{(n)} \rangle_{U_n} = \langle \pi(\chi^s), \text{Ind}^G_{U_n}(\psi_{(n)}) \rangle, \]

so \( \Psi = \Psi \circ \pi \).

(e) Induct on \( n \), using (c) and the identity

\[ (-1)^{n-1}p_n(Y^{(1)}) = ne_n(Y^{(1)}) - \sum_{r=1}^{n-1}(-1)^{r-1}p_r(Y^{(1)})e_{n-r}(Y^{(1)}), \quad [\text{Ma, I.2.11}] \]

to obtain \( \tilde{\Psi}(p_n(Y^{(1)})) = 1 \). Note that

\[ \tilde{\Psi}(p_n(Y^{(1)})) = \tilde{\Psi}(\pi(p_n(Y^{(1)}))) = \tilde{\Psi}((-1)^{\varphi|n-1}p_{\varphi|n}(X^{(t-1)})) = \tilde{\Psi}(\pi(p_{\varphi|k}(Y^{(1)}))) = \tilde{\Psi}(p_{\varphi|k}(Y^{(1)})) = 1 = \tilde{\Psi}(p_k(Y^{(1)})) \]

Since \( \tilde{\Psi} \) is multiplicative on \( \Lambda_C(Y^{(1)}) \),

\[ \tilde{\Psi}(p_{\nu}(Y^{(1)})) = 1 = \tilde{\Psi}(p_{\nu}(Y^{(1)})), \quad \text{for all partitions } \nu. \]
In particular, since \( \tilde{\Psi} \) is linear and \( \Lambda_C(Y(\varphi)) = \mathbb{C}\text{-span}\{p_\nu(Y(\varphi))\} \),
\[
\tilde{\Psi}(f(Y(\varphi))) = \tilde{\Psi}(f(Y(1))), \quad \text{for all } f \in \Lambda_C(Y(\varphi)).
\]

Note that (e) also implies that \( \tilde{\Psi} \) is multiplicative on all of \( \Lambda_C \).

(f) Note that
\[
\tilde{\Psi}(s_\nu(Y(\varphi))) = \tilde{\Psi}(f(Y(\varphi))) = \tilde{\Psi}(f(Y(1))),
\]
for all \( f \in C(Y(\varphi)) \).

Note that (e) also implies that \( \tilde{\Psi} \) is multiplicative on all of \( \Lambda_C(Y(\varphi)) \).

(f) Note that
\[
\tilde{\Psi}(s_\nu(Y(\varphi))) = \tilde{\Psi}(f(Y(\varphi))) = \tilde{\Psi}(f(Y(1))),
\]
for all \( f \in C(Y(\varphi)) \).

Decomposition of \( \text{Ind}^G_U(\psi_\mu) \). Suppose \( \lambda, \nu \in \mathcal{P}^\Theta \). A column strict tableau \( P \) of shape \( \lambda \) and weight \( \nu \) is a column strict filling of \( \lambda \) such that for each \( \varphi \in \Theta \),
\[
\text{sh}(P^\varphi) = \lambda^\varphi \quad \text{and} \quad \text{wt}(P^\varphi) = \nu^\varphi.
\]

We can now prove the theorem stated at the beginning of this section:

**Theorem 5.1 ([Ze])** Let \( U \) be the subgroup of unipotent upper-triangular matrices of \( G = \text{GL}_n(F_q) \), \( \mu \models n \) and \( \psi_\mu \) be as in (3.1). Then
\[
\text{Ind}^G_U(\psi_\mu) = \bigoplus_{\lambda \in \mathcal{H}_\mu} (G^\lambda)^{\text{det}^\lambda_{\mathcal{H}_\mu}}.
\]

**Proof.** Note that
\[
\text{Ind}^P_U(\psi_\mu) \cong \bigoplus_{\mu \in \mathcal{P}^\Theta} (C\text{-span})\{e_\mu \}
\]
where
\[
e_{[\mu]} = \frac{1}{|U \cap L_\mu|} \sum_{u \in U \cap L_\mu} \psi_\mu(u^{-1})u \quad \text{and} \quad e'_{[\mu]} = \frac{1}{|U_\mu|} \sum_{u \in U_\mu} u.
\]

Thus,
\[
\text{Ind}^P_U(\psi_\mu) \cong \bigoplus_{\mu \in \mathcal{P}^\Theta} (C\text{-span})\{e_\mu \}
\]
where
\[
e_{[\mu]} = \frac{1}{|U \cap L_\mu|} \sum_{u \in U \cap L_\mu} \psi_\mu(u^{-1})u \quad \text{and} \quad e'_{[\mu]} = \frac{1}{|U_\mu|} \sum_{u \in U_\mu} u.
\]

Thus,
\[
\text{Ind}^P_U(\psi_\mu) \cong \bigoplus_{\mu \in \mathcal{P}^\Theta} (C\text{-span})\{e_\mu \}
\]
where
\[
e_{[\mu]} = \frac{1}{|U \cap L_\mu|} \sum_{u \in U \cap L_\mu} \psi_\mu(u^{-1})u \quad \text{and} \quad e'_{[\mu]} = \frac{1}{|U_\mu|} \sum_{u \in U_\mu} u.
\]

In particular, by the definition of multiplication in \( R (5.1) \),
\[
\Gamma_\mu = \text{ch} (\text{Ind}^G_U(\psi_\mu)) = \Gamma_{\mu_1} \Gamma_{\mu_2} \cdots \Gamma_{\mu_t}, \quad \text{where} \quad \Gamma_{\mu_t} = \sum_{\lambda \in \mathcal{P}^\Theta, \text{ht}(\lambda) = 1} s_\lambda.
\]
Pieri’s rule (2.7) implies that for \( \lambda \in \mathcal{P}_r^\Theta, \nu \in \mathcal{P}_s^\Theta \) and \( \text{ht}(\nu) = 1, \)

\[
 s_\lambda s_\nu = \sum_{\gamma \in \mathcal{P}_r^\Theta, |\mathcal{H}_{1/\lambda}^\gamma| \neq 0} s_\gamma, \quad \text{so} \quad \Gamma_{\mu} = \sum_{\lambda \in \mathcal{P}_r^\Theta} K_{\lambda\mu} s_\lambda,
\]

where

\[
 K_{\lambda\mu} = \text{Card}\{\emptyset \supseteq \gamma_0 \subseteq \gamma_1 \subseteq \gamma_2 \subseteq \cdots \subseteq \gamma_\ell = \lambda : |\mathcal{H}_{(\mu_i+1)}^{\gamma_{i+1}/\gamma_i}| = 1\} = \text{Card}\{\text{column strict tableaux of shape } \lambda \text{ and weight } \mu\} = |\mathcal{H}_\mu^{\lambda}|.
\]

By Green’s Theorem (Theorem 5.2), \( ch \) is an isomorphism, so

\[
 \text{Ind}_U^G(\psi_\mu) = \text{ch}^{-1}(\Gamma_\mu) = \sum_{\lambda \in \mathcal{R}_\mu} |\mathcal{H}_\mu^{\lambda}| \text{ch}^{-1}(s_\lambda) = \sum_{\lambda \in \mathcal{R}_\mu} (G^\lambda)^{\oplus |\mathcal{H}_\mu^{\lambda}|}. \quad \square
\]

6 A weight space decomposition of \( \mathcal{H}_\mu \)-modules

Let \( \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \models n \) and let \( P_\mu, L_\mu \) and \( U_\mu \) be as in (2.10) and (2.11). Recall that

\[
 e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1})u.
\]

Theorem 6.1. For \( a \in M_\mu \), let \( T_a = e_\mu v_a e_\mu \) with \( v_a \) as in (3.6). Then the map

\[
 \mathcal{H}_{(\mu_1)} \otimes \mathcal{H}_{(\mu_2)} \otimes \cdots \otimes \mathcal{H}_{(\mu_\ell)} \
 \xrightarrow{T_{(f_1)} \otimes T_{(f_2)} \otimes \cdots \otimes T_{(f_\ell)}} \mathcal{H}_\mu \
 \text{for } (f_i) \in M_\mu,
\]

is an injective algebra homomorphism with image \( \mathcal{L}_\mu = e_\mu P_\mu e_\mu = e_\mu L_\mu e_\mu. \)

Proof. Note that

\[
 T_{(f_1)} \otimes T_{(f_2)} \otimes \cdots \otimes T_{(f_\ell)} = \frac{1}{|U \cap L_\mu|^2} \sum_{x_i, y_i \in U_\mu} \left( \prod_{i=1}^\ell \psi_\mu(x_i^{-1} y_i^{-1}) \right) x_1 v_{(f_1)} y_1 \otimes x_2 v_{(f_2)} y_2 \otimes \cdots \otimes x_\ell v_{(f_\ell)} y_\ell.
\]

Since \( U = (L_\mu \cap U)(U_\mu), \ L_\mu \cap U \cong U_{\mu_1} \times U_{\mu_2} \times \cdots \times U_{\mu_\ell}, \) and \( \psi_\mu \) is trivial on \( U_\mu, \)

\[
 T_{(f_1)} \otimes (f_2) \otimes \cdots \otimes (f_\ell) = \frac{1}{|U|^2} \sum_{x, y \in U} \psi_\mu(x^{-1} y^{-1}) x(v_{(f_1)} \oplus v_{(f_2)} \oplus \cdots \oplus v_{(f_\ell)}) y
 = \frac{1}{|U \cap L_\mu|^2} \sum_{x_i, y_i \in U_\mu} \psi_\mu(x_i^{-1} y_i^{-1} \oplus \cdots \oplus x_\ell^{-1} y_\ell^{-1}) e_\mu(x_1 v_{(f_1)} y_1 \oplus \cdots \oplus x_\ell v_{(f_\ell)} y_\ell),
\]

where \( e_\mu \) is as in (5.6). Since \( L_\mu \subseteq N_G(U_\mu), \) the idempotent \( e_\mu \) commutes with \( x_1 v_{(f_1)} y_1 \oplus \cdots \oplus x_\ell v_{(f_\ell)} y_\ell \) and

\[
 T_{(f_1)} \otimes (f_2) \otimes \cdots \otimes (f_\ell) = \frac{e_\mu}{|L \cap U|^2} \sum_{x_i, y_i \in U_\mu} \left( \prod_{i=1}^\ell \psi_\mu(x_i^{-1} y_i^{-1}) \right) x_1 v_{(f_1)} y_1 \oplus \cdots \oplus x_\ell v_{(f_\ell)} y_\ell.
\]

Consequently, the map multiplies by \( e_\mu \) and changes \( \otimes \) to \( \oplus, \) so it is an algebra homomorphism. Since the map sends basis elements to basis elements, it is also injective. \( \square \)
Let $\mathcal{L}_\mu$ be as in Theorem 6.1. By Theorem 3.1 each $\mathcal{H}(\mu_i)$ is commutative, so $\mathcal{L}_\mu$ is commutative and all the irreducible $\mathcal{L}_\mu$-modules $\mathcal{L}_\mu^\gamma$ are one-dimensional. Theorem 4.2 implies that
\[
\mathcal{H}(\mu_i) = \{ \Theta \text{-partitions } \lambda : |\lambda| = \mu_i, \text{ht}(\lambda) = 1 \}.
\]
indexes the irreducible $\mathcal{H}(\mu_i)$-modules. Therefore, the set
\[
\hat{\mathcal{L}}_\mu = \hat{\mathcal{H}}(\mu_1) \times \hat{\mathcal{H}}(\mu_2) \times \cdots \times \hat{\mathcal{H}}(\mu_\ell) = \{ \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell) : \gamma_i \in \hat{\mathcal{H}}(\mu_i) \}
\]
indexes the irreducible $\mathcal{L}_\mu$-modules. Identify $\cap$ with the map $\gamma : \mathcal{L}_\mu \rightarrow \mathbb{C}$ such that
\[
yv = \gamma(y)v, \quad \text{for all } y \in \mathcal{L}_\mu, v \in \mathcal{L}_\mu^\gamma.
\]
For $\gamma \in \hat{\mathcal{L}}_\mu$, define the $\gamma$-weight space $V_\gamma$ of an $\mathcal{H}_\mu$-module $V$ to be
\[
V_\gamma = \{ v \in V : yv = \gamma(y)v, \text{ for all } y \in \mathcal{L}_\mu \}.
\]
Then
\[
V \cong \bigoplus_{\gamma \in \hat{\mathcal{L}}_\mu} V_\gamma.
\]

Let $\lambda \in \mathcal{P}^\Theta$ and $\gamma \in \hat{\mathcal{L}}_\mu$. A column strict tableau $P$ of shape $\lambda$ and weight $\gamma$ is column strict filling of $\lambda$ such that for each $\varphi \in \Theta$,
\[
\text{sh}(P^{(\varphi)}) = \lambda^{(\varphi)} \quad \text{and} \quad \text{wt}(P^{(\varphi)}) = (|\gamma_1^{(\varphi)}|, |\gamma_2^{(\varphi)}|, \ldots, |\gamma_\ell^{(\varphi)}|),
\]
where $|\gamma_i^{(\varphi)}|$ is the number of boxes in the partition $\gamma_i^{(\varphi)}$ (which has length 1).

**Theorem 6.2.** Let $\mathcal{H}_\mu^\lambda$ be an irreducible $\mathcal{H}_\mu$-module and $\gamma \in \hat{\mathcal{L}}_\mu$. Then
\[
\dim(\mathcal{H}_\mu^\lambda)_\gamma = \text{Card}\{ \text{column strict tableaux of shape } \lambda \text{ and weight } \gamma \} = |\hat{\mathcal{H}}_\lambda^\gamma|.
\]

**Proof.** By double-centralizer theory and Frobenius reciprocity,
\[
\dim((\mathcal{H}_\mu^\lambda)_\gamma) = (\text{Res}_{\mathcal{L}_\mu}^{\mathcal{H}_\mu}(\mathcal{H}_\mu^\lambda), \mathcal{L}_\mu^\gamma) = (\text{Res}_{P_\mu}^G(G^\lambda), P_\mu^\gamma) = (G^\lambda, \text{Ind}_{P_\mu}^G(P_\mu^\gamma)),
\]
where $P_\mu^\gamma = \text{Inf}_{\mathcal{L}_\mu}^{P_\mu}(L_\mu^\gamma)$. Therefore,
\[
\dim((\mathcal{H}_\mu^\lambda)_\gamma) = c_\gamma^\lambda, \quad \text{where } s_{\gamma_1}s_{\gamma_2} \cdots s_{\gamma_\ell} = \sum_{\lambda \in \mathcal{P}^\Theta} c_\lambda^\lambda s_\lambda.
\]
Pieri’s rule (2.7) implies $c_\gamma^\lambda = |\hat{\mathcal{H}}_\lambda^\gamma|$. \hfill \Box

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