Problem (11.2.3). Let $V$ be the collection of polynomials with coefficients in $F$ in the variable $x$ of degree at most $n$. Determine the transition matrix from the basis $1, x, x^2, \ldots, x^n$ for $V$ to the elements

$$1, x - \lambda, \ldots, (x - \lambda)^{n-1}, (x - \lambda)^n$$

where $\lambda$ is a fixed element of $F$. Conclude that these elements are a basis for $V$.

Solution. Since $x = (x - \lambda) + \lambda$, by the binomial theorem, we have

$$x^k = ((x - \lambda) + \lambda)^k = \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i}(x - \lambda)^i.$$ 

Hence, the desired transition matrix is given by

$$
\begin{pmatrix}
1 & \lambda & \lambda^2 & \ldots & \lambda^n \\
0 & 1 & 2\lambda & \ldots & n\lambda^{n-1} \\
0 & 0 & 1 & \ldots & (n)\lambda^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
$$

Since this matrix is invertible, $1, x - \lambda, \ldots, (x - \lambda)^n$ are also a basis for $V$.

Problem (11.2.4). Let $\varphi$ be the linear transformation of $\mathbb{R}^2$ to itself given by rotation counterclockwise around the origin through an angle $\theta$. Show that the matrix of $\varphi$ with respect to the standard basis for $\mathbb{R}^2$ is

$$
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
$$

Solution. Let $M$ be the matrix of $\varphi$ with respect to the standard basis. Then the first column of $M$ is $\varphi(1, 0)$, and the second column is $\varphi(0, 1)$. Since $\varphi(1, 0) = (\cos \theta, \sin \theta)$ and $\varphi(0, 1) = (-\sin \theta, \cos \theta)$, we have

$$M = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.$$ 

Problem (11.2.5). Show that the $m \times n$ matrix $A$ is nonsingular if and only if the linear transformation $\varphi$ is a nonsingular linear transformation from the $n$-dimensional space $V$ to the $m$-dimensional space $W$, where $A = M_{\mathcal{E}}^\mathcal{B}(\varphi)$, regardless of the choice of bases $\mathcal{B}$ and $\mathcal{E}$.

Date: 31 January, 2011.
Solution. Suppose $A = M^B_\varphi$, where $\varphi$ is a nonsingular linear transformation. Then $\varphi$ has an inverse $\varphi^{-1}$, and $A^{-1} = M^B_{\varphi^{-1}}$, since the composite of the two transformations is the identity, and the matrix is taken to be from $B$ to $B$ or $E$ to $E$ (depending on the order of composition). Conversely, if $A$ is nonsingular, then $A^{-1} = M^B_{\varphi^{-1}}$.

Problem (11.2.9). If $W$ is a subspace of the vector space $V$ stable under the linear transformation $\varphi$ (i.e., $\varphi(W) \subseteq W$), show that $\varphi$ induces linear transformations $\varphi|_W$ on $W$ and $\overline{\varphi}$ on the quotient vector space $V/W$. If $\varphi|_W$ and $\overline{\varphi}$ are nonsingular prove $\varphi$ is nonsingular. Prove the converse holds if $V$ has finite dimension and give a counterexample with $V$ infinite dimensional.

Solution. Let $w \in W$. Then, since $W$ is stable under $\varphi$, $\varphi(w) \in W$, so $\varphi|_W$ induces a linear transformation on $W$. Now, suppose $v \in V$. We define $\overline{\varphi}(v+W) = \varphi(v) + W$. We must check that this is well-defined. Suppose $v' - v = w \in W$. Then

$$\overline{\varphi}(v') - \overline{\varphi}(v) = \varphi(v') + W - \varphi(v) + W = \varphi(v' - v) + W = \varphi(w) + W = W,$$

so $\overline{\varphi}$ is well-defined. (It is easily checked that $\varphi$ is in fact a linear transformation.) Suppose that $\varphi|_W$ and $\overline{\varphi}$ are nonsingular. We'll check first that $\varphi$ is surjective. Pick $v \in V$. Since $\overline{\varphi}$ is surjective, we can find some $v' \in V$ so that $\overline{\varphi}(v' + W) = v + W$; in other words, $\varphi(v) = v' + w$ for some $w \in W$. Since $\varphi|_W$ is surjective, we can find some $w' \in W$ so that $\varphi(w') = w$. Hence $\varphi(v' - w') = v$. Now, we check that $\varphi$ is injective. Suppose $\varphi(v) = 0$. Hence $\overline{\varphi}(v + W) = W$, we must have $v \in W$. Since $\varphi|_W$ is injective, and we have $\varphi|_W(v) = 0$, we must have $v = 0$.

Now, we check the converse for a finite-dimensional space $V$. Suppose $\varphi$ is nonsingular. Then $\varphi$ is injective, so $\varphi|_W$ is also injective and hence nonsingular (since $W$ is finite-dimensional). Now we show that $\overline{\varphi}$ is surjective. Pick $v' + W \in V/W$. Since $\varphi$ is surjective, there is some $v \in V$ so that $\varphi(v) = v'$. Hence $\overline{\varphi}(v + W) = v' + W$, so $\overline{\varphi}$ is surjective. For a counterexample in the infinite-dimensional space, let $V$ be a vector space with basis $\{e_i : i \in \mathbb{Z}\}$, and let $\varphi$ be the transformation given by the right shift: $\varphi(e_i) = e_{i+1}$. Let $W$ be the subspace of $V$ generated by $\{e_i : i \geq 0\}$. Hence $W$ is a $\varphi$-invariant subspace, but $\overline{\varphi}$ on $V/W$ fails to be injective: since $\varphi(e_{-1}) = e_0$, $\overline{\varphi}(e_{-1} + W) = e_0 + W = W$. Furthermore, $\varphi|_W$ fails to be surjective, since $e_0 \not\in \text{image } \varphi|_W$.

Problem (11.2.10). Let $V$ be an $n$-dimensional vector space and let $\varphi$ be a linear transformation of $V$ to itself. Suppose $W$ is a subspace of $V$ of dimension $m$ that is stable under $\varphi$.

(a) Prove that there is a basis for $V$ with respect to which the matrix for $\varphi$ is of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$
where $A$ is an $m \times m$ matrix, $B$ is an $m \times (n - m)$ matrix and $C$ is an $(n - m) \times (n - m)$ matrix (such a matrix is called block upper triangular).

(b) Prove that there is a subspace $W'$ invariant under $\varphi$ so that $V = W \oplus W'$ decomposes as a direct sum then the bases for $W$ and $W'$ give a basis for $V$ with respect to which the matrix for $\varphi$ is block diagonal:

$$
\begin{pmatrix}
A & 0 \\
0 & C
\end{pmatrix}
$$

where $A$ is an $m \times m$ matrix and $C$ is an $(n - m) \times (n - m)$ matrix.

(c) Prove conversely that if there is a basis for $V$ with respect to which $\varphi$ is block diagonal as in (b) then there are $\varphi$-invariant subspaces $W$ and $W'$ of dimensions $m$ and $n - m$, respectively, with $V = W \oplus W'$.

**Solution.**

(a) Let $\{e_1, \ldots, e_m\}$ be a basis for $W$, and complete this to a basis $\{e_1, \ldots, e_n\}$ of $V$. Since $W$ is stable under $\varphi$, $\varphi(e_i) \in W$ for $1 \leq i \leq m$, so the matrix of $\varphi$ with respect to this basis is of the desired form.

(b) In this case, let $\{e_1, \ldots, e_m\}$ be a basis of $W$, and let $\{e_{m+1}, \ldots, e_n\}$ be a basis for $W'$. Then the matrix of $\varphi$ with respect to the basis $\{e_1, \ldots, e_n\}$ is of the form $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$, where $A$ is an $m \times m$ matrix and $C$ is an $(n - m) \times (n - m)$ matrix.

(c) Suppose such a basis exists; call it $\{e_1, \ldots, e_n\}$. Then, let $W$ be the span of $e_1, \ldots, e_m$, and let $W'$ be the span of $e_{m+1}, \ldots, e_n$. Then $W$ and $W'$ are both $\varphi$-stable of the correct dimensions, and $V = W \oplus W'$.

**Problem (11.2.11).** Let $\varphi$ be a linear transformation from the finite-dimensional vector space $V$ to itself such that $\varphi^2 = \varphi$.

(a) Prove that image $\varphi \cap \ker \varphi = 0$.

(b) Prove that $V = \text{image} \varphi \oplus \ker \varphi$.

(c) Prove that there is a basis of $V$ such that the matrix of $\varphi$ with respect to this basis is a diagonal matrix whose entries are all 0 or 1.

**Solution.**

(a) Suppose $v \in V$ be such that $\varphi(v) \neq 0$. Then $\varphi^2(v) = \varphi(v) \neq 0$, so $\varphi(v) \notin \ker \varphi$. Hence image $\varphi \cap \ker \varphi = 0$.

(b) Let $v \in V$. We write $v = v' + v''$, where $v' \in \text{image} \varphi$ and $v'' \in \ker \varphi$. Let $v' = \varphi(v)$, and let $v'' = v - \varphi(v)$. Then clearly $v' \in \text{image} \varphi$, and

$$
\varphi(v'') = \varphi(v) - \varphi(\varphi(v)) = \varphi(v) - \varphi(v) = 0,
$$

so $v'' \in \ker \varphi$. By part (a), then, $V = \text{image} \varphi \oplus \ker \varphi$.

(c) Let $\{e_1, \ldots, e_m\}$ be a basis of image $\varphi$, and let $\{e_{m+1}, \ldots, e_n\}$ be a basis of $\ker \varphi$. Then the matrix of $\varphi$ with respect to this basis is a diagonal matrix all of whose diagonal entries are 0 or 1.
Problem (11.2.37). Let $V$ be the 7-dimensional vector space over the field $F$ consisting of the polynomials in the variable $x$ of degree at most 6. Let $\varphi$ be the linear transformation of $V$ to itself defined by $\varphi(f) = f'$, where $f'$ denotes the usual derivative (with respect to $x$) of the polynomial $f \in V$. For each of the fields below, determine a basis for the image and for the kernel of $\varphi$:

(a) $F = \mathbb{R}$.
(b) $F = \mathbb{F}_2$, the finite field of 2 elements (note that, for example, $(x^2)' = 2x = 0$ over this field).
(c) $F = \mathbb{F}_3$.
(d) $F = \mathbb{F}_5$.

Solution. (a) The image consists of all polynomials of degree at most 5, so a basis for the image is $\{1, x, x^2, x^3, x^4, x^5\}$, and a basis for the kernel is $\{1\}$.

(b) When we differentiate a monomial $x^n$ of even degree, we get 0, so the image consists of polynomials all of whose monomials have even degree (and degree less than 6). Hence, a basis for the image is $\{1, x^2, x^4\}$, and a basis for the kernel is $\{1, x^2, x^4, x^6\}$.

(c) This case is similar to the one above. Now a basis for the image is $\{1, x, x^3, x^4\}$, and a basis for the kernel is $\{1, x^3, x^6\}$.

(d) This time, a basis for the image is $\{1, x, x^2, x^3, x^5\}$, and a basis for the kernel is $\{1, x^5\}$.

Problem (11.3.1). Let $V$ be a finite dimensional vector space. Prove that the map $\varphi \mapsto \varphi^*$ in Theorem 20 gives a ring isomorphism of $\text{End}(V)$ with $\text{End}(V^*)$.

Solution. This is false: the map is not a ring homomorphism in general (when $\dim V > 1$): we have

$$ (\varphi \psi)^*(f) = f \circ (\varphi \circ \psi) $$
$$ = (f \circ \varphi) \circ \psi $$
$$ = \psi^*(f \circ \varphi) $$
$$ = \psi^*(\varphi^* f) $$
$$ = \psi^* \varphi^*(f). $$

But, we can at least show that the map is a vector space isomorphism. In Theorem 20, we show that $\varphi \mapsto \varphi^*$ is a vector space homomorphism. We now show that the map is an isomorphism. Since $V$ and $V^*$ have the same dimension, $\text{End}(V)$ and $\text{End}(V^*)$ also have the same dimension. Hence, it suffices to show that $\varphi \mapsto \varphi^*$ is injective. Suppose that $\varphi \neq 0$. Then, there is some $v \in V$ so that $\varphi(v) \neq 0$. Hence, there is some $f \in V^*$ so that $f(\varphi(v)) \neq 0$. In particular, $f \circ \varphi$ is not the zero element of $V^*$, so $\varphi^*(f) \neq 0$. Hence $\varphi^* \neq 0$. Thus $\varphi \mapsto \varphi^*$ is injective and hence an isomorphism.

Problem (11.3.2). Let $V$ be the collection of polynomials with coefficients in $\mathbb{Q}$ in the variable $x$ of degree at most 5 with $1, x, x^2, \ldots, x^5$ as basis. Prove that the following
are elements of the dual space of $V$ and express them as linear combinations of the dual basis:

(a) $E : V \to \mathbb{Q}$ defined by $E(p(x)) = p(3)$ (i.e., evaluation at $x = 3$).

(b) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = \int_0^1 p(t) \, dt$.

(c) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = \int_0^1 t^2 p(t) \, dt$.

(d) $\varphi : V \to \mathbb{Q}$ defined by $\varphi(p(x)) = p'(5)$ where $p'$ denotes the usual derivative of the polynomial $p(x)$ with respect to $x$.

**Solution.** Let $e_0 = 1, \ldots, e_5 = x^5$ be a basis of $V$, so the dual basis is $e_0^*, \ldots, e_5^*$.

(a) $(f + g)(3) = f(3) + g(3)$, so $E$ is an element of $V^*$. To determine it in terms of the dual basis, note that $E(e_i) = 3^i$ for $0 \leq i \leq 5$, so $E = e_0^* + 3e_1^* + 9e_2^* + 27e_3^* + 81e_4^* + 243e_5^*$.

(b) Again, integration is linear, so $\varphi$ is an element of $V^*$. Checking the image on the basis, we see that $\varphi = e_0^* + \frac{1}{2}e_1^* + \frac{1}{3}e_2^* + \frac{1}{4}e_3^* + \frac{1}{5}e_4^* + \frac{1}{6}e_5^*$.

(c) This time, we have $\varphi = \frac{1}{3}e_0^* + \frac{1}{4}e_1^* + \frac{1}{5}e_2^* + \frac{1}{6}e_3^* + \frac{1}{7}e_4^* + \frac{1}{8}e_5^*$.

(d) Differentiation is also linear, so $\varphi \in V^*$. We have $\varphi = 5e_1^* + 10e_2^* + 15e_3^* + 20e_4^* + 25e_5^*$.

**Problem (1).** Let $A$ denote an $m \times n$ matrix over a field $k$. Prove that there are invertible matrices $P$ and $Q$ over $k$, with $P$ (respectively $Q$) of size $m \times m$ (respectively $n \times n$), so that $PAQ$ has the form

$$
\begin{bmatrix}
I_p & 0_{p \times (n-p)} \\
0_{(m-p) \times p} & 0_{(m-p) \times (n-p)}
\end{bmatrix}
$$

where $p$ is a number which is less than or equal to both $m$ and $n$, $I_p$ denotes the $p \times p$ identity matrix, and $0_{a \times b}$ denotes the $a \times b$ zero matrix.

**Solution.** Given a matrix $A$, we can perform row and column operations to put it into the desired form. The matrices $P$ and $Q$ keep track of the row and column operations, respectively, that were performed.

**Problem (2).** Determine such matrices $P$ and $Q$, as well as the integer $p$, for the matrix

$$
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
-3 & -2 & -1
\end{bmatrix}.
$$
Solution. We row-reduce the matrix first to get $P$, keeping track of the operations performed. For example, we may begin by subtracting twice the first row from the second and adding three times the first row to the third; this corresponds to left-multiplication by
\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
3 & 0 & 1
\end{bmatrix}.
\]
When we finish row-reducing, we get
\[
P = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ -5 & 4 & 1 \end{bmatrix}, \quad PA = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Now, we column-reduce to get $Q$. When we do this, we find that
\[
Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad PAQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Hence, $p = 2$. 