Problem (10.4.8). Suppose \( R \) is an integral domain with quotient field \( Q \) and let \( N \) be any \( R \)-module. Let \( U = R^\times \) be the set of nonzero elements in \( R \) and define \( U^{-1}N \) to be the set of equivalence classes of ordered pairs of elements \((u, n)\) with \( u \in U \) and \( n \in N \) under the equivalence relation \((u, n) \sim (u', n')\) if and only if \( u'n = un' \) in \( N \).

(a) Prove that \( U^{-1}N \) is an abelian group under the operation defined by \( (u_1, n_1) + (u_2, n_2) = (u_1u_2, u_2n_1 + u_1n_2) \). Prove that \( r(u, n) = (u, rn) \) defines an action of \( R \) of \( U^{-1}N \) making it into an \( R \)-module. (This is an example of localization considered in general in Section 4 of Chapter 15, cf. also Section 5 in Chapter 7.)

(b) Show that the map from \( Q \times N \) to \( U^{-1}N \) defined by sending \((a/b, n)\) to \((b, an)\) for \( a \in R \), \( b \in U \), \( n \in N \), is an \( R \)-balanced map, so induces a homomorphism \( f \) from \( Q \otimes_R N \) to \( U^{-1}N \). Show that the map \( g \) from \( U^{-1}N \) to \( Q \otimes_R N \) defined by \( g((u, n)) = (1/u) \otimes n \) is well-defined and is an inverse homomorphism to \( f \). Conclude that \( Q \otimes_R N \cong U^{-1}N \) as \( R \)-modules.

(c) Conclude from (b) that \((1/d) \otimes n \) is 0 in \( Q \otimes_R N \) if and only if \( rn = 0 \) for some nonzero \( r \in R \).

(d) If \( A \) is an abelian group, show that \( Q \otimes_Z A = 0 \) if and only if \( A \) is a torsion abelian group (i.e., every element of \( A \) has finite order).

Solution.  

(a) It is straightforward to check that letting the identity element be \((1, 0)\) and the inverse of \((u, n)\) be \((u, -n)\) gives an abelian group structure on \( U^{-1}N \), and that the suggested \( R \)-action in fact does make \( U^{-1}N \) into an \( R \)-module.

(b) The only interesting thing to check in regard to the \( R \)-balancedness is that, if we call the map \( \varphi \), then \( \varphi(a/b, n) + \varphi(c/d, n) = \varphi(a/b + c/d, n) \). We have

\[
\varphi(a/b, n) + \varphi(c/d, n) = \frac{b}{an} + \frac{d}{cn} = \frac{bd}{an} + \frac{bcn}{an} = \varphi((ad + bc)/(bd), n) = \varphi(a/b + c/d, n),
\]

as desired. Hence, \( \varphi \) induces a map \( Q \otimes_R N \to U^{-1}N \). The rest is again straightforward.
(c) Suppose $rn = 0$ for some nonzero $r \in R$. We have

$$(1/d) \otimes n = g((d,n)) = g((rd, rn)) = (1/rd) \otimes (rn) = \frac{1}{rd} \otimes 0 = 0,$$

as desired. Now, if $$(1/d) \otimes n = 0,$$

then we have

$$0 = (1/d) \otimes n = 1 \otimes dn = g((1,dn)),$$

so $dn = 0$.

(d) We know from part (c) that the torsion elements of $A$ are exactly those that “become” zero in $Q \otimes Z A$. Hence $Q \otimes Z A = 0$ if and only if $A$ is torsion.

**Problem (10.4.9).** Suppose $R$ is an integral domain with quotient field $Q$ and let $N$ be any $R$-module. Let $Q \otimes_R N$ be the module obtained from $N$ by extension of scalars from $R$ to $Q$. Prove that the kernel of the $R$-module homomorphism $\iota : N \rightarrow Q \otimes_R N$ is the torsion submodule of $N$ (cf. Exercise 8 in Section 1). [Use the previous exercise.]

**Solution.** The proof is very similar to that of 10.4.8(c).

**Problem (10.4.13).** Prove that the usual dot product of vectors defined by letting $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n)$ be $a_1 b_1 + \cdots + a_n b_n$ is a bilinear map from $R^n \times R^n$ to $R$.

**Solution.** We have to show that $(r_1 a_1 + r_2 a_2) \cdot b = r_1 (a_1 \cdot b) + r_2 (a_2 \cdot b)$. This is either easily checked by hand or else recalled from elementary linear algebra.

**Problem (10.4.16).** Suppose $R$ is commutative and let $I$ and $J$ be ideals of $R$, so $R/I$ and $R/J$ are naturally $R$-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \mod I) \otimes (r \mod J)$.

(b) Prove that there is an $R$-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \mod I) \otimes (r' \mod J)$ to $rr' \mod (I + J)$.

**Solution.** (a) We’ll first check that a simple tensor can be rewritten in the desired form. A typical simple tensor is of the form $(r \mod I) \otimes (s \mod J)$, but

$$(r \mod I) \otimes (s \mod J) = r(1 \mod I) \otimes (s \mod J) = (1 \mod I) \otimes (rs \mod J).$$

Now, suppose instead that we have a sum of simple tensors, which looks like

$$\sum (r_i \mod I) \otimes (s_i \mod J).$$

By the above, this is equal to

$$\sum (1 \mod I) \otimes (r_i s_i \mod J) = (1 \mod I) \otimes \left(\sum r_i s_i \mod J\right),$$

which is a simple tensor of the desired form.
(b) Let \( \varphi : R/I \otimes R/J \to R/(I + J) \) be the map in the problem, and let \( \psi : R/(I + J) \to R/I \otimes R/J \) be given by \( \psi(r \mod (I + J)) = (1 \mod I) \otimes (r \mod J) \). These are both homomorphisms; let’s show that they are inverses. We have

\[
\psi \circ \varphi((r \mod I) \otimes (r \mod J)) = \psi(rr' \mod (I + J)) = (1 \mod I) \otimes (rr' \mod J);
\]

by part (a), this is equal to \( (r \mod I) \otimes (r' \mod J) \), so this composition is the identity. For the other direction, we have

\[
\varphi \circ \psi(r \mod (I + J)) = \varphi((1 \mod I) \otimes (r \mod J)) = r \mod (I + J),
\]

so this composition is also the identity. Hence \( \varphi \) is an isomorphism.

**Problem (10.4.24).** Prove that the extension of scalars from \( \mathbb{Z} \) to the Gaussian integers \( \mathbb{Z}[i] \) of the ring \( \mathbb{R} \) is isomorphic to \( \mathbb{C} \) as a ring: \( \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C} \) as rings.

**Solution.** We’ll check that \( \mathbb{C} \) satisfies the universal property for the tensor product. This means that, if we let \( \psi : \mathbb{Z}[i] \times \mathbb{R} \to \mathbb{C} \) be the tensor product map \( \psi(a + bi, r) = ar + bri \) and \( \varphi : \mathbb{Z}[i] \times \mathbb{R} \to A \) is a bilinear map, then there’s a unique map \( \theta : \mathbb{C} \to A \) so that \( \theta \circ \psi = \varphi \). We define \( \theta(\alpha + \beta i) = \varphi(1,a) + \varphi(i,b) \). Hence, for \( a + bi \in \mathbb{Z}[i] \) and \( r \in \mathbb{R} \), we have

\[
\theta \circ \psi(a + bi, r) = \theta(ar + bri) = \varphi(1, ar) + \varphi(i, br) = a\varphi(1, r) + b\varphi(i, r) = \varphi(a + bi, r),
\]

as desired. In fact, this only shows that \( \psi \) is an isomorphism **of modules**, but since all the maps in sight can be taken to be ring maps, the isomorphism is in fact an isomorphism of rings.

Well, perhaps there’s a bit of a story to be told in that last remark, actually. Whenever we deal with universal properties, we’re talking about categorical notions. These are generally not too heavily dependent in which category we’re working in. In this case, we begin by talking about the tensor product of two modules, but now we’re suddenly discussing the tensor product of two **algebras** (or rings). The reason we can move so easily between these two seemingly different notions is that their definitions are essentially identical: when all our maps are only required to be module maps, we find that certain things are isomorphic as modules. When we require them to be ring maps, our arguments generally go through just as before to show that they’re isomorphic as rings.
Problem (10.4.27). (a) Write down a formula for the multiplication of two elements \(a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4\) and \(a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4\) in the example \(A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}\) following Proposition 21 (where \(1 = 1 \otimes 1\) is the identity of \(A\)).

(b) Let \(\varepsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)\) and \(\varepsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)\). Show that \(\varepsilon_1 \varepsilon_2 = 0\), \(\varepsilon_1 + \varepsilon_2 = 1\), and \(\varepsilon_j^2 = \varepsilon_j\) for \(j = 1, 2\) (\(\varepsilon_1\) and \(\varepsilon_2\) are called *orthogonal idempotents* in \(A\).)

Deduce that \(A\) is isomorphic as a ring to the direct product of two principal ideals: \(A \cong A\varepsilon_1 \times A\varepsilon_2\) (cf. Exercise 1, Section 7.6).

(c) Prove that the map \(\varphi : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}\) by \(\varphi(z_1, z_2) = (z_1 z_2, z_1 \overline{z}_2)\), where \(\overline{z}_2\) denotes the complex conjugate of \(z_2\), is an \(\mathbb{R}\)-bilinear map.

(d) Let \(\Phi\) be the \(\mathbb{R}\)-module homomorphism from \(A\) to \(\mathbb{C} \times \mathbb{C}\) obtained from \(\varphi\) in (c). Show that \(\Phi(\varepsilon_1) = (0, 1)\) and \(\Phi(\varepsilon_2) = (1, 0)\). Show also that \(\Phi\) is \(\mathbb{C}\)-linear, where the action of \(\mathbb{C}\) is on the left tensor factor in \(A\) and on both factors in \(\mathbb{C} \times \mathbb{C}\). Deduce that \(\Phi\) is surjective. Show that \(\Phi\) is a \(\mathbb{C}\)-algebra isomorphism.

Solution. (a) We have

\[
(a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4) \\
\times (a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4) = (aa' - bb' - cc' + dd' \cdot 1 \\
\quad + (ab' + ba' - cd' - dc') \cdot e_2 \\
\quad + (ac' + ca' - bd' - db') \cdot e_3 \\
\quad + (ad' + da' - bc' - cb') \cdot e_4.
\]

(b) The first part follows directly from the formula above. The fact that rings decompose in this way is a general property: the map \(A \rightarrow A\varepsilon_1 \times A\varepsilon_2\) is given by \(a \mapsto (a \varepsilon_1, a \varepsilon_2)\).

(c) Just check it.

(d) We have

\[
\Phi(1 \otimes 1) = \varphi(1, 1) = (1, 1),
\]

\[
\Phi(i \otimes i) = \varphi(i, i) = (-1, 1),
\]

so \(\Phi(\varepsilon_1) = \frac{1}{2}((1, 1) + (-1, 1)) = (0, 1)\) and \(\Phi(\varepsilon_2) = \frac{1}{2}((1, 1) - (-1, 1)) = (1, 0)\).

To show that \(\Phi\) is \(\mathbb{C}\)-linear, we compute

\[
\Phi(z(z_1 \otimes z_2)) = \Phi(zz_1 z_2, z z_1 \overline{z}_2) = (z z_1 z_2, z z_1 \overline{z}_2),
\]

as desired. Now, \(\Phi\) is surjective because it’s \(\mathbb{C}\)-linear, and its image contains a basis of the codomain. Finally, it’s an isomorphism because it’s a surjective map of \(\mathbb{C}\)-vector spaces of the same (finite) dimension.

Problem (1). Let \(\mathbb{Z}[X]\) denote the ring of polynomials with coefficients in the integers, and let \(\mathbb{Z}[X^2]\) denote the subring of all polynomials in \(X^2\). Let \(\mathbb{Z}\) denote the \(\mathbb{Z}[X^2]\)-module whose underlying group is \(\mathbb{Z}\) and where \(X^2\) acts by \(X^2 \cdot n = 0\) for all \(n\).
Describe the abelian group structure of $\mathbb{Z}[X] \otimes_{\mathbb{Z}[X^2]} \mathbb{Z}$, i.e. provide a decomposition of abelian groups

$$\mathbb{Z}[X] \otimes_{\mathbb{Z}[X^2]} \mathbb{Z} \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$$

for some choice of $r$ and integers $n_i$.

**Solution.** We claim that $\mathbb{Z}[X] \otimes_{\mathbb{Z}[X^2]} \mathbb{Z} \cong \mathbb{Z}^2$. To do this, we'll show that $\mathbb{Z}^2$ satisfies the universal property for the tensor product. Let $\rho : \mathbb{Z}[X] \times \mathbb{Z} \to \mathbb{Z}^2$ be the $\mathbb{Z}[X^2]$-bilinear map defined by $\rho(\sum a_i x^i, n) = (a_1 n, a_0 n)$. Now, let $\varphi : \mathbb{Z}[X] \times \mathbb{Z} \to A$ be any $\mathbb{Z}[X^2]$-bilinear map. We need to show that there's a unique map $\psi : \mathbb{Z}^2 \to A$ so that $\rho \circ \psi = \varphi$. The only possibility is to set $\psi(a, b) = \varphi(ax + b, 1)$. Hence, $\mathbb{Z}^2$ satisfies the desired universal property.

There was a fair amount of magic in that solution: we had to guess, somehow, that $\mathbb{Z}^2$ fit there, and then we had to define the map $\rho$ carefully. We can, in fact, do this problem a bit more systematically, using some properties of tensor products of polynomial rings, as follows: $\mathbb{Z}[X] \cong \mathbb{Z}[X^2, Y]/(Y^2 - X^2)$, and $\mathbb{Z} \cong \mathbb{Z}[X^2]/(X^2)$. Hence we have

$$\mathbb{Z}[X] \otimes_{\mathbb{Z}[X^2]} \mathbb{Z} \cong \mathbb{Z}[X^2, Y]/(Y^2 - X^2) \otimes_{\mathbb{Z}[X^2]} \mathbb{Z}[X^2]/(X^2)$$
$$\cong \mathbb{Z}[X^2, Y]/(Y^2 - X^2, X^2)$$
$$\cong \mathbb{Z}[X^2, Y]/(Y^2, X^2)$$
$$\cong \mathbb{Z}[Y]/(Y^2)$$
$$\cong \mathbb{Z}^2.$$

**Problem (2).** Let $\mathbb{Z}_{\mathbb{Z}^p}$ denote the abelian group of rational numbers whose denominators are powers of the prime $p$. Determine whether the group

$$\mathbb{Z} \left[ \frac{1}{p} \right] \otimes_{\mathbb{Z}} \mathbb{Z} \left[ \frac{1}{p} \right] / \mathbb{Z}$$

is zero or nonzero. Prove your assertion.

**Solution.** The group is zero. It suffice to check that any simple tensor is zero. Let $(a/p^m) \otimes (b/p^n)$ be a simple tensor. (The second term should really be $b/p^n$ mod $\mathbb{Z}$, but we'll be sloppy with our notation.) We have

$$(a/p^m) \otimes (b/p^n) = (ap^n/p^{m+n}) \otimes (b/p^n)$$
$$= p^n(a/p^{m+n}) \otimes (b/p^n)$$
$$= (a/p^{m+n}) \otimes (bp^n/p^n)$$
$$= (a/p^{m+n}) \otimes b$$
$$= (a/p^{m+n}) \otimes 0$$
$$= 0.$$
Hence, any simple tensor is zero, so the entire group is also zero.

This problem should be considered in analogy with Problem 10.4.8 above. In that problem, we saw that when we tensor with $\mathbb{Q}$, we annihilate all the torsion. Here, we saw that when we tensor with $\mathbb{Z}[1/p]$, then we annihilate $p$-power torsion: every element of $\mathbb{Z}[1/p]/\mathbb{Z}$ is $p^n$ torsion for some $n$. More generally, if we tensor with a ring in which a prime $q$ is invertible, we annihilate all $q$-power torsion. Hence, in the case of $\mathbb{Q}$ in which every prime is invertible, we annihilate torsion at all primes.

**Problem (3).** Prove the isomorphism $k[x] \otimes_k k[y] \cong k[x, y]$ for any field $k$.

**Solution.** We have a $k$-bilinear map $\rho: k[x] \times k[y] \to k[x, y]$ given by $(p(x), q(y)) \mapsto p(x)q(y)$. Let $\theta: k[x] \times k[y] \to A$ be any $k$-bilinear map. We must show that there is a unique $k$-module (or vector space) homomorphism $\varphi: k[x, y] \to A$ so that $\rho \circ \varphi = \theta$. It suffices to define $\varphi$ on monomials of the form $x^iy^j$, since we can then extend it by linearity. We define $\varphi(x^iy^j) = \theta(x^i, y^j)$; this clearly makes the diagram commute. Furthermore, this is the only option to make the desired relation hold, since $\rho(x^i, y^j) = x^iy^j$. Hence, $k[x, y]$ satisfies the universal property for the tensor product, so we have $k[x] \otimes_k k[y] \cong k[x, y]$, as desired.

**Problem (4).** Describe the $\mathbb{Q}$-vector space $\mathbb{Q}[x]/(x) \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x^2 + 1)$, i.e. compute its dimension.

**Solution.** By Problem 10.4.16 above, we have

$$
\frac{\mathbb{Q}[x]}{(x)} \otimes_{\mathbb{Q}[x]} \frac{\mathbb{Q}[x]}{(x^2 + 1)} \cong \frac{\mathbb{Q}[x]}{(x, x^2 + 1)} \cong \frac{\mathbb{Q}[x]}{\mathbb{Q}[x]} \cong 0.
$$