1 30.6

Recall that both the rationals and irrationals are dense in $\mathbb{R}$. Pick $a \in \mathbb{R}$. Then, there exists a sequence $x_j \in \mathbb{R}$ so that $\lim_{j \to \infty} x_j = a$ and for $j$ even, $x_j \in \mathbb{Q}$ and for $j$ odd, $x_j \not\in \mathbb{Q}$.

Taking $\epsilon = \frac{1}{2}$. If the limit existed, then there would exist $\delta > 0$ so that for $|y - a| < \delta$, then $|f(y) - f(a)| < \frac{1}{2}$. By convergence of the sequence $x_j$, there is $J$ so that $|x_j - a| < \delta$ for $j > J$. Taking $j > J$ even or odd, depending on whether or not $a \in \mathbb{Q}$, we may arrange that $|f(a) - f(x_j)| = 1$ for some $j > J$, a contradiction.

2 34.3

If $f$ is not bounded on $[a, b]$, then there exists $\{a_n\}$ in $[a, b]$ so that $|f(a_n)| > n$. By the Bolzano–Weierstrass theorem, we may extract a convergent subsequence, which we continue to denote $\{a_n\}$. Let $\lim_{n \to \infty} a_n = c$. Then, by continuity of $f$, we have that $f(c) = \lim_{n \to \infty} f(a_n)$. However, this is a contradiction: it would imply that for $\epsilon = 1$, there exists $N$ so that for $n > N$, we have that

$$1 > |f(c) - f(a_n)| \geq |f(a_n)| - |f(c)| \geq n - |f(c)|,$$

by the triangle inequality. This is a contradiction for $n$ sufficiently large.

3 34.4

We first claim that

$$\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right) = (0, 1).$$

To see this, note that any $x \in (0, 1)$ is bigger than $\frac{1}{n}$ for $n$ sufficiently large. On the other hand, because the sets $\left(\frac{1}{n}, 1\right)$ are nested, if $\mathcal{U}$ is a finite subcollection, letting $M$ denote the largest $n$ appearing in $\mathcal{U}$, it is clear that

$$\bigcup_{U \in \mathcal{U}} U = \left(\frac{1}{M}, 1\right).$$

This does not contain $(0, 1)$.

4 34.5

Suppose that $\{x_n\}$ is a bounded sequence. Then it is contained in some bounded interval $[a, b]$. Assume for contradiction that for all $p \in [a, b]$ there is an open interval $U_p \ni p$ so that $U_p$ contains only a finite number of elements of $\{x_n\}$. Clearly $\{U_p : p \in [a, b]\}$ is an open cover of $[a, b]$. By the Heine–Borel property, we may find a finite subcover $U_{p_1}, U_{p_2}, \ldots, U_{p_M}$. Because each set contains at most a finite number of elements of $\{x_n\}$ and there are a finite number of sets, there must be a finite number of elements of $\{x_n\}$ in $\bigcup_{j=1}^{M} U_{p_j} \supset [a, b]$. This is a contradiction, because there are an infinite number of elements in a sequence.

Hence, there exists $p \in [a, b]$ so that any open interval containing $p$ contains an infinite number of elements of $\{x_n\}$. We choose $U := \left(p - \frac{1}{j}, p + \frac{1}{j}\right)$. Choose a subsequence $x_{n_j}$ so that $x_{n_j} \in U_j$. To do so, we use the fact
that each $U_j$ contains an infinite number of elements of $x_n$: this lets us choose an index $n_j$ which is bigger than $n_{j-1}$. It is clear that $x_{n_j}$ converges to $p$.

5 34.6

Fix $\epsilon > 0$. By continuity of $f$, for each point $p \in [a, b]$, there is $\delta_p > 0$ so that if $x \in (p - 2\delta_p, p + 2\delta_p)$ then $|f(x) - f(p)| < \frac{\epsilon}{2}$. The collection $\{(p - \delta_p, p + \delta_p) : p \in [a, b]\}$ is an open cover, so we may find a finite subcover $(p_1 - \delta_{p_1}, p_1 + \delta_{p_1}), \ldots, (p_M - \delta_{p_M}, p_M + \delta_{p_M})$. Let $\delta = \min\{\delta_{p_1}, \ldots, \delta_{p_M}\}$.

Suppose that $x, y \in [a, b]$ satisfy $|x - y| < \delta$. Because we have found a subcover, there is some $j$ so that $x \in (p_j - \delta_{p_j}, p_j + \delta_{p_j})$. By the triangle inequality, $|x - y| < \delta$ and $\delta \leq \delta_{p_j}$ we see that also

$$y \in (p_j - 2\delta_{p_j}, p_j + 2\delta_{p_j}).$$

Now, by choice of $\delta_p$, we have that $|f(x) - f(p_j)| < \frac{\epsilon}{2}$ and $|f(y) - f(p_j)| < \frac{\epsilon}{2}$, so the triangle inequality implies that

$$|f(x) - f(y)| < \epsilon.$$

6 35.3

This follows from the triangle inequality:

$$d(x, z) \leq d(x, y) + d(x, z),$$

which implies that

$$d(x, z) - d(x, y) \leq d(x, z).$$

Similarly, the triangle inequality implies that

$$d(x, y) \leq d(x, z) + d(y, z),$$

so

$$d(x, y) - d(z, y) \leq d(x, z).$$

These combine to give the desired statement.

7 35.6

7.1 a

By Theorem 22.3, if $\{a_n\}$ in $l^1$, then $\lim_{n \to \infty} a_n = 0$, so $\{a_n\} \in c_0$. If $\{a_n\} \in c_0$, then there is some $N$ sufficiently large so that $|a_n - 0| < 1$ for $n \geq N$. Hence,

$$\sup_{n \geq 1} |a_n| \leq \max \left\{ \sup_{1 \leq n < N} |a_n|, 1 \right\} < \infty.$$ 

This proves the desired inclusions.

7.2 b

To see that the first inclusion $l^1 \subset c_0$ is not proper, consider $\{1/n\}$, which is in $c_0$ but not $l^1$. To see that the second inclusion $c_0 \subset l^\infty$ is not proper, consider $\{1, 1, 1, 1, \ldots\}$. 

2
7.3 c

The triangle inequality for absolute value implies that \( d'(\{a_n\}, \{b_n\}) < \infty \) for \( \{a_n\}, \{b_n\} \in l^\infty \). We now check the properties of a norm: (i) Clearly \( d'(\{a_n\}, \{a_n\}) = 0 \). Suppose that \( d'(\{a_n\}, \{b_n\}) = 0 \). Then \( \sup |a_n - b_n| = 0 \). This can only happen if \( a_n = b_n \) for all \( n \), i.e., \( \{a_n\} = \{b_n\} \). (ii) for \( \{a_n\}, \{b_n\} \in l^\infty \), we have that

\[
d'(\{a_n\}, \{b_n\}) = \sup_n |a_n - b_n| = \sup_n |b_n - a_n| = d'(\{b_n\}, \{a_n\}).
\]

Finally, for \( \{a_n\}, \{b_n\}, \{c_n\} \in l^\infty \), we have that

\[
d'(\{a_n\}, \{b_n\}) = \sup_n |a_n - b_n| \leq \sup_n (|a_n - c_n| + |c_n - b_n|) \leq \sup_n |a_n - c_n| + \sup_n |b_n - c_n| = d'(\{a_n\}, \{c_n\}) + d'(\{b_n\}, \{c_n\}).
\]

Here, we have used the property that

\[
sup(\gamma_n + \eta_n) \leq sup \gamma_n + sup \eta_n,
\]

which is proven in Theorem 20.6.

Hence \( d'(\cdot, \cdot) \) is a norm on \( l^\infty \).

7.4 d

For any \( \{a_n\}, \{b_n\} \in l^1 \) and \( \epsilon > 0 \), we choose \( m \) so that

\[
d'(\{a_n\}, \{b_n\}) - |a_m - b_m| < \epsilon.
\]

Then,

\[
d'(\{a_n\}, \{b_n\}) < |a_m - b_m| + \epsilon \leq \epsilon + \sum_{n=1}^\infty |a_n - b_n| = \epsilon + d(\{a_n\}, \{b_n\}).
\]

Because \( \epsilon > 0 \) was arbitrary, this implies that

\[
d'(\{a_n\}, \{b_n\}) \leq d(\{a_n\}, \{b_n\}).
\]

8

First, we show that \( f : \mathcal{F} \to [0,1] \) is 1-1. Suppose that there are two sets \( A^{(1)}, A^{(2)} \in \mathcal{F} \) so that \( f(A^{(1)}) = f(A^{(2)}) \). Choose

\[
k_n^{(j)} := \begin{cases} 
1 & n \in A^{(j)} \\
0 & n \notin A^{(j)}
\end{cases},
\]

for \( j = 0,1 \). Then, it is clear that

\[
f(A^{(j)}) = \sum_{n=1}^\infty 2^{-n}k_n^{(j)}.
\]

Let \( M \in \mathbb{N} \) denote some upper bound of \( A^{(1)} \cup A^{(2)} \). Then, by assumption

\[
\sum_{n=1}^M 2^{-n}k_n^{(1)} = \sum_{n=1}^M 2^{-n}k_n^{(2)}
\]

Multiplying by \( 2^M \) and defining the index \( m = M - n \), we see that

\[
\sum_{m=0}^{M-1} 2^m k_{M-m}^{(1)} = \sum_{n=1}^M 2^m k_{M-m}^{(2)}
\]

These are two base 2 representations of the same number. Hence, by uniqueness of base 2 representations, we see that \( k_{M-m}^{(1)} = k_{M-m}^{(2)} \), i.e. that \( A^{(1)} = A^{(2)} \).
Clearly the image of \( f : \mathcal{F} \to [0, 1] \) are the rational numbers in \([0, 1]\) which can be written as \( \frac{p}{2^r} \) for \( p, M \in \mathbb{N} \). To check this, note that any element of the image is of this form, by forming a common denominator in the sum, which must necessarily be a power of two. Conversely, for \( p, M \in \mathbb{N} \), we may find \( k_{M-n} \) so that

\[
\sum_{m=0}^{M-1} 2^m k_{M-m} = p.
\]

That we can find such a sequence \( k_{M-n} \) follows because it must hold that \( p < 2^M \). Now, we see that \( f(\{k_n\}) = \frac{p}{2^r} \), by the same computation as before.

\section*{9}

For \( x \in [0, 1] \) define \( k_n \) inductively as follows: \( k_n = 1 \) exactly when

\[
2^{-n} + \sum_{j=1}^{n-1} 2^{-j} k_j \leq x.
\]

Otherwise, set \( k_n = 0 \). By construction, the partial sums defining \( f(\{k_n\}) \) are all \( \leq x \). Hence, we see that

\[
f(\{k_n\}) \leq x.
\]

On the other hand, we claim that

\[
\sum_{j=1}^{n} 2^{-j} k_j \geq x - 2^{-n},
\]

which is easily proven by induction, using the definition of \( k_n \). This implies that \( f(\{k_n\}) = x \).

Finally, we investigate the points where \( f \) fails to be 1-1. Suppose that \( f(\{k_n^{(1)}\}) = f(\{k_n^{(2)}\}) \). Choose the largest \( N \) so that \( k_n^{(1)} = k_n^{(2)} \) for \( n < N \). Because \( k_N^{(1)} \neq k_N^{(2)} \), switching the sequences if necessary, we may arrange that \( k_N^{(1)} = 1 \) and \( k_N^{(2)} = 0 \). Now, we have that

\[
0 = f(\{k_n^{(1)}\}) - f(\{k_n^{(2)}\}) = 2^{-N} + \sum_{n=N+1}^{\infty} 2^{-n} (k_n^{(1)} - k_n^{(2)}).
\]

Because \( |k_n^{(1)} - k_n^{(2)}| \leq 1 \), we have that

\[
\sum_{n=N+1}^{\infty} 2^{-n} (k_n^{(2)} - k_n^{(1)}) \leq 2^{-N}
\]

with equality if and only if \( k_n^{(2)} - k_n^{(1)} = 1 \) for all \( n \geq N + 1 \). Hence, \( k_n^{(2)} = 1, k_n^{(1)} = 0 \) for \( n \geq N + 1 \). As such, \( f \) fails to be 1-1 exactly at points of the form

\[
k_1, k_2, \ldots, k_{N-1}, 1, 0, 0, 0, \ldots
\]

\[
k_1, k_2, \ldots, k_{N-1}, 0, 1, 1, 1, \ldots
\]

\section*{10}

Because \( f : \mathcal{S} \to [0, 1] \) is surjective, \( \mathcal{S} \) must be uncountable (as \([0, 1]\) is uncountable). On the other hand, because \( f \) is a bijection between \( \mathcal{F} \) and the rationals in \([0, 1]\) of the form \( \frac{p}{2^r} \), which is clearly a countable set, we see that \( \mathcal{F} \) is countable.