Homework # 2.

Problem 1. Consider the Poisson equation in a bounded domain $U$:
\[ -\Delta u = f(x) \text{ in } U, \]
with $f(x) \geq 0$. Use Problem 5 of Problem set 2 to show that $u(x)$ can not attain a minimum inside $U$.

Problem 2. Consider the Poisson equation in a bounded domain $U$:
\[ -\Delta u = f(x) \text{ in } U. \]
However, instead of prescribing the value of $u$ at the boundary we prescribe its normal derivative:
\[ \frac{\partial u}{\partial n} = g \text{ on } \partial U. \]

(i) Show that if $u(x)$ solves the problem (2)-(3) then so does any function of the form $u(x) + C$, where $C$ is an arbitrary constant.

(ii) Show that for (2)-(3) to have a solution we must have
\[ \int_U f(x)dx = -\int_{\partial U} g(y)dS(y). \]

(iii) Fix $x \in U$ and let $h(x; y)$ be the solution of
\[ -\Delta_y h(x, y) = 0 \text{ for } y \in U, \]
\[ \frac{\partial h(x, y)}{\partial n_y} = \frac{\partial \Phi(x - y)}{\partial n_y} + \frac{1}{S}. \]

Here $\Phi(x)$ is the fundamental solution of the Laplace equation, and $S = |\partial U|$ is the area of the boundary $\partial U$. Show that constraint (4) is satisfied for (5), hence (5) may have a solution.

(iv) Let $h(x, y)$, with $x, y \in U$, be a solution of (5). Set
\[ N(x, y) = \Phi(x - y) - h(x, y), \text{ for } x, y \in U. \]
Adapt our derivation for the problem when the boundary data for $u$ (rather than for its normal derivative) is prescribed to show that the function
\[ u(x) = \int_U N(x, y)f(y)dy + \int_{\partial U} N(x, y)g(y)dS(y) \]
gives a solution for (2)-(3).

Problem 3. Let a function $u$ satisfy the following problem:
\[ \Delta^2 u = 0 \text{ in } U, \]
\[ \Delta u = g \text{ on } \partial U, \]
\[ u = 0 \text{ on } \partial U. \]

Here $\Delta^2$ is the bi-Laplacian, that is, $\Delta^2 u = \Delta(\Delta u)$. Here we prescribe the values of $u$ and its Laplacian on the boundary – but the equation inside $U$ is now of the fourth order.
(i) Compute the bi-Laplacian in two dimensions explicitly in terms of the fourth order derivatives of \( u \):

\[
\Delta^2 u = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 u = ?
\]

(ii) Let \( u \) solve (6), and set \( h = \Delta u \). Show that \( h \) attains its minimum and maximum over \( U \) at the boundary.

(iii) Assume that \( g \geq 0 \). Show that \( u \) attains its maximum over \( U \) at the boundary. Is this true about the minimum as well?

(iv) Show that (6) has at most one solution.

**Problem 4.** Consider the integer lattice \( \mathbb{Z}^2 \) in two dimensions, and let \( X_n(x, y) \) be the standard random walk starting at a point \( (x, y) \) on the lattice (\( n \) here is the number of jumps made by the walk). Let \( D \) be a domain on the lattice and \( \partial D \) its boundary. Use the arguments similar to those in the lecture notes for the discrete Laplace equation in order to find the probabilistic interpretation for the discrete Poisson problem

\[
-u(x+1, y) - u(x-1, y) - u(x, y+1) - u(x, y-1) + 4u(x, y) = f(x, y) \text{ in } D,
\]

with the boundary condition \( u(x, y) = 0 \) if \( (x, y) \in \partial D \). Pass to the continuum limit and obtain (formally, with no proof) the probabilistic interpretation for the solution of the Poisson problem posed in a two-dimensional domain \( \Omega \),

\[
-\Delta u = f(x, y), \quad (x, y) \in \Omega,
\]

with the boundary condition \( u = 0 \) on \( \partial \Omega \).