Problem 1.

Define \( v(t, x) = u(t, x + bt) \), then

\[
\frac{\partial v}{\partial t}(t, x) - a(x + bt) \frac{\partial^2 u}{\partial x^2}(t, x) = 0, \forall t > 0, x \in \mathbb{R},
\]

\[ v(0, x) = f(x). \]

It suffices to show that \( v(t, x) \leq F = \max_{y \in \mathbb{R}} f(y) \).

The main difficulty is that \( U_T = \{ (t, x) : 0 < t \leq T \} \) is not a bounded domain. So it’s not enough to show that there’s no local maximum point. (Note that there’s no assumption that, for fixed \( t \), \( u(t, x) \to 0 \) as \( x \to \infty \). So the behavior of \( u(t, \pm \infty) \) needs to be examined.) To deal with this, we consider the domain \( U_{T,L} = \{ (t, x) : 0 < t \leq T, -L < x < L \} \), where \( T > 0, L > 0 \). And we consider the auxiliary function:

\[
w(t, x) = v(t, x) - F - \frac{M}{L^2}(2\bar{a}t + x^2),
\]

where \( \bar{a} = 10 > a(x) \), \( F = \sup_{y \in \mathbb{R}} f(y) \), and \( M = \sup_{t \geq 0, x \in \mathbb{R}} v(t, x) \) + 1 (since \( u(t, x) \) is a bounded solution, \( M \) is a well defined real positive number).

We have

\[
\frac{\partial w}{\partial t}(t, x) - a(x + bt) \frac{\partial^2 w}{\partial x^2}(t, x) = -\frac{2M}{L^2}(\bar{a} - a(x + bt)) < 0, \forall (t, x) \in U_{T,L},
\]

\[
w(0, x) \leq 0, \forall x \in (-L, L),
\]

\[
w(t, \pm L) \leq 0, \forall t \in [0, T].
\]

By a similar reasoning as in Theorem 1.16 in the notes, we know that \( w(t, x) \) must attain its maximum over \( U_{T,L} \) on the parabolic boundary \( \Gamma_{T,L} = \{ (t, x) : t = 0 \land -L < x < L \lor 0 \leq t \leq T \land x = \pm L \} \).

The reasoning is:

Suppose \( w(t, x) \) attains its maximum at \( (t_0, x_0) \in U_{T,L} \), then

\[
\frac{\partial w}{\partial t}(t_0, x_0) \geq 0, \frac{\partial^2 w}{\partial x^2}(t_0, x_0) \leq 0,
\]

\[
\frac{\partial w}{\partial t}(t_0, x_0) - a(x) \frac{\partial^2 w}{\partial x^2}(t_0, x_0) \geq 0,
\]

which is a contradiction to \( \frac{\partial w}{\partial t}(t_0, x_0) - a(x) \frac{\partial^2 w}{\partial x^2}(t_0, x_0) < 0 \). Thus \( w \) must attain its maximum on \( \Gamma_{T,L} \).
(The key point is that, we transform the unbounded domain $U_T$ to a bounded domain $U_{T,L}$ so the similar technique can be used. For the domain $U_T$, this reasoning cannot be used directly, because the behavior of the function when $x$ approaches $\infty$ needs to be examined.)

So we have that $w(t, x) \leq 0, \forall x \in \bar{U}_{T,L}$, i.e.,

$$v(t, x) \leq F + \frac{M}{L^2} (2\bar{a}t + x^2), \forall t \in [0, T], x \in [-L, L].$$

Now, for any point $(t_1, x_1) \in \mathbb{R}^+ \times \mathbb{R}$, we want to show that $v(t_1, x_1) \leq F$. Fist we set $T = t_1$, and then we choose $L > |x_1|$. Then the conclusion we obtained can be used for $v(t_1, x_1)$, i.e.,

$$v(t_1, x_1) \leq F + \frac{M}{L^2} (2\bar{a}t_1 + x_1^2).$$

Now, note that it’s true for any $L > |x_1|$. Let $L \to \infty$ and we have

$$v(t_1, x_1) \leq F.$$

**Problem 2.**

(i) By Fourier transform we have

$$\frac{\partial \hat{u}}{\partial t}(t, k) + 4\pi^2 k^2 \hat{u}(t, k) = 0, \forall t > 0, k \in \mathbb{R},$$

$$\hat{u}(0, k) = \hat{f}(k).$$

Solve the equation we have

$$\hat{u}(t, k) = e^{-4\pi^2 k^2 t} \hat{f}(k),$$

$$u(t, x) = \int \hat{u}(t, k)e^{2\pi i k x} dk = \int e^{2\pi i k x - 4\pi^2 k^2 t} \hat{f}(k) dk.$$

(ii)

$$u(t, x) = \int e^{2\pi i k x - 4\pi^2 k^2 t} \hat{f}(k) dk$$

$$= \int e^{2\pi i k x - 4\pi^2 k^2 t} \int e^{-2\pi i k y} f(y) dy dk$$

$$= \int \left( \int e^{2\pi i k (x - y) - 4\pi^2 k^2 t} dk \right) f(y) dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int \left( \int e^{-\pi \xi^2} e^{2\pi i ((x - y)/\sqrt{4\pi t})} d\xi \right) f(y) dy \quad (\xi = \sqrt{4\pi t} k)$$

$$= \frac{1}{\sqrt{4\pi t}} \int e^{-\pi ((x - y)/\sqrt{4\pi t})^2} f(y) dy$$

$$= \int G(t, x - y) f(y) dy$$

2
Problem 3.

(i) 
\[ \hat{f}_0 = \int_0^1 \left( \frac{1}{2} - x \right) \, dx = 0. \]
\[ \hat{f}_n = \int_0^1 e^{-2\pi inx} \left( \frac{1}{2} - x \right) \, dx = \left( -\frac{e^{-2\pi inx}}{4\pi in} + \frac{xe^{-2\pi inx}}{2\pi in} - \frac{e^{-2\pi inx}}{4\pi^2 n^2} \right) \bigg|_{x=0}^{x=1} = \frac{1}{2\pi in}, \forall n \neq 0. \]

(ii) 
\[ (S_N f)'(x) = \left( \sum_{k=-N}^{N} \hat{f}_k e^{2\pi ikx} \right)' = \sum_{k=-N}^{N} (2\pi ik\hat{f}_k) e^{2\pi ikx} = D_N(x) - 1, \]
and 
\[ S_N f(0) = \sum_{k=-N}^{N} \hat{f}_k = 0, \]
thus 
\[ S_N f(x) = \int_0^x (D_N(t) - 1) \, dt = \int_0^x \frac{\sin((2N+1)t)}{\sin(\pi t)} \, dt - x. \]

(iii) 
\[ S_N f(x) - \tilde{S}_N f(x) = \int_0^x \left( \frac{1}{\sin(\pi t)} - \frac{1}{\pi t} \right) \sin((2N+1)\pi t) \, dt. \]

Define 
\[ g(z) = \frac{1}{\sin(z)} - \frac{1}{z}, \]
we have
\[ g(z) = \frac{1}{\sin(z)} - \frac{1}{z} = \frac{1}{z - z^3/3! + z^5/5! - z^7/7! + \ldots} - \frac{1}{z} \]
\[ = \frac{z^2/3! - z^4/5! + z^6/7! + \ldots}{z - z^3/3! + z^5/5! - z^7/7! + \ldots} \]
\[ = \frac{z/3! - z^3/5! + z^5/7! + \ldots}{1 - z^2/3! + z^4/5! - z^6/7! + \ldots} \]
so 
\[ z = 0 \]
is a removable singular point. We define 
\[ g(0) = 0, \]
then 
\[ g(z) \]
is a \( C^\infty \) function on \([0, \pi/4]\), then 
\[ h(t) := \frac{1}{\sin(\pi t)} - \frac{1}{\pi t} \in C^\infty[0, 1/4], \]
then

$$\forall x \in [0, 1/4],$$
\[
\left| \int_0^x \left( \frac{1}{\sin(\pi t)} - \frac{1}{\pi t} \right) \sin((2N + 1)\pi t) \, dt \right| = \left| \int_0^x h(t) \sin((2N + 1)\pi t) \, dt \right|
\]
\[
= \left| - \frac{h(t) \cos((2N + 1)\pi t)}{(2N + 1)\pi} \right|_{t=0}^{t=x} + \left| \int_0^x \frac{h'(t) \cos((2N + 1)\pi t)}{(2N + 1)\pi} \, dt \right|
\]
\[
\leq \frac{C_1 + C_2}{(2N + 1)\pi},
\]

where

$$C_1 = 2 \sup_{t \in [0, 1/4]} |h(t)|, C_2 = \int_0^{1/4} |h'(t)| \, dt. \quad \text{Defining } C = C_1 + C_2,$$

then

$$\left| \int_0^x h(t) \sin((2N + 1)\pi t) \, dt \right| \leq C/N, \forall x \in [0, 1/4].$$

(iv)

$$\tilde{S}_N f(x_N) = -1/(2N) + \int_0^{1/(2N)} \frac{\sin((2N + 1)\pi t)}{\pi t} \, dt$$
\[
= -1/(2N) + \frac{1}{\pi} \int_0^{(2N + 1)\pi/(2N)} \frac{\sin(y)}{y} \, dy \quad (y = (2N + 1)\pi t)
\]
\[
\Rightarrow \quad \lim_{N \to \infty} \tilde{S}_N f(x_N)
\]
\[
= \lim_{N \to \infty} \left( -1/(2N) + \frac{1}{\pi} \int_0^{(2N + 1)\pi/(2N)} \frac{\sin(y)}{y} \, dy \right)
\]
\[
= \frac{1}{\pi} \int_0^{\pi} \frac{\sin(y)}{y} \, dy.
\]

(v) Consider the function

$$r(x) := \frac{\pi - x}{\pi}, p(x) =: \frac{\sin(x)}{x}.$$

What we want to prove is that

$$\frac{1}{\pi} \int_0^\pi p(x) \, dx > 1/2.$$
Obviously, we have
\[ \frac{1}{\pi} \int_{0}^{\pi} r(x) \, dx = 1/2 \]
we just need to prove \( p(x) \neq r(x) \) (which is obvious) and \( p(x) \geq r(x) \) (\( \forall x \in [0, \pi] \)):
First, we have
\[
p(x) - r(x) = \frac{\sin(x)}{x} - \frac{\pi - x}{\pi} = \frac{\pi \sin(x) - x(\pi - x)}{\pi x}.
\]
Now we consider the function \( q(x) := \pi \sin(x) - x(\pi - x) \). We need to show that \( q(x) \geq 0 \), \( \forall x \in [0, \pi/2] \). Because of the symmetry of \( q(x) \) (which means that \( q(x) = q(\pi - x), \forall x \in [0, \pi] \)), we just need to show \( q(x) \geq 0, \forall x \in [0, \pi/2] \). Note that \( q(0) = 0, q'(x) = \pi \cos(x) - \pi + 2x \), \( q'(x) \) is concave on \([0, \pi/2]\), and \( q'(0) = q'(\pi/2) = 0 \), we have
\[
q'(x) \geq 0, \forall x \in [0, \pi/2],
q(x) \geq 0, \forall x \in [0, \pi/2],
q(x) \geq 0, \forall x \in [0, \pi],
p(x) \geq r(x), \forall x \in [0, \pi],
\]
\( \bar{S} > 1/2 \).

This means, as \( N \) goes to infinity, there exists \( x_N = 1/(2N) \), which goes to 0, such that \( S_N f(x_N) \) goes to some constant which is strictly greater than 1/2. But we want to use \( S_N f(x) \) to approximate \( f(x) \), and we know that \( |f(x)| \leq 1/2 \). So the sequence \( S_N f(x) \) does not converge to \( f(x) \) uniformly. This is mainly because \( f(x) \) is not continuous at \( x = 0 \) (after we extend \( f \) periodically). This is the Gibbs phenomenon. If we want to use \( S_N f(x) \) as an approximation of \( f(x) \), then near \( x = 0 \), we need \( N \) to be large to get a relatively good approximation.

**Problem 4.**

We have \( v_{xx} + v(1 - v^2) = 0 \). Multiply by \( v_x \) on both sides we have
\[
v_x v_{xx} + v(1 - v^2)v_x = 0,
(v_x^2/2)_x + (v^2/2)_x - (v^4/4)_x = 0,
v_x^2 + v^2 - v^4/2 = C_1,
v_x^2 = (1 - v^2)^2/2 + C_2,
\]
where \( C_2 = C_1 - 1/2 \).

Note that \( v(-\infty) = -1 \), \( v(+\infty) = 1 \) and \(-1 < v(x) < 1 \), we have
\[
\lim_{x \to \infty} v_x(x)^2 = C_2.
\]
But \( v(+\infty) = 1 \) exists, so \( C_2 = 0 \), and
\[
v_x = \pm (1 - v^2)/\sqrt{2}.
\]
Since \( v(\pm \infty) > v(\mp \infty) \), we should choose the branch

\[
v_x = (1 - v^2)/\sqrt{2}.
\]

Then we have

\[
\frac{dv}{1 - v^2} = dx/\sqrt{2},
\]

\[
\log \frac{1 + v}{1 - v} = \sqrt{2}(x - C_3),
\]

\[
v(x) = \frac{e^{\sqrt{2}(x-C_3)} - 1}{e^{\sqrt{2}(x-C_3)} + 1} = \tanh \left( \frac{x - C_3}{\sqrt{2}} \right),
\]

where \( C_3 \) is an arbitrary number.

If the equation is

\[
\varepsilon v_{xx} + v(1 - v^2) = 0,
\]

then by a similar calculation we have

\[
v(x) = \tanh \left( \frac{x - C_3}{\sqrt{2\varepsilon}} \right).
\]

As \( \varepsilon \to 0 \), we have

\[
v(x) \to \text{sgn}(x - C_3) = \begin{cases} 
-1, & \forall x < C_3, \\
0, & \text{if } x = C_3, \\
1, & \forall x > C_3.
\end{cases}
\]