Homework # 5

1. (i) Let $a_k$ be the Fourier coefficients of the function $u(x) = 1/2 - x$, $0 < x < 1$, extended periodically:

$$a_k = \int_0^1 \left(\frac{1}{2} - x\right) e^{-2\pi ikx} \, dx.$$ 

Show that there exists a constant $M > 0$ so that for all $x \in [0, 1]$ and all $N \in \mathbb{Z}$ the partial sums of the Fourier series satisfy

$$\left| \sum_{k=-N}^{N} a_k e^{2\pi ikx} \right| \leq M.$$

(ii) Let two integer sequences $N_k \geq 1$ and $m_k \geq 1$ be such that

$$N_{k+1} - m_{k+1} > N_k + m_k,$$

and define

$$f(x) = \sum_{k=1}^{\infty} e^{2\pi i N_k x} \frac{B_k(x)}{k^2},$$

with

$$B_k(x) = \sum_{j=-m_k}^{m_k} a_j e^{2\pi ijx}.$$ 

Show that the function $f(x)$ is continuous and find its Fourier coefficients in terms of $a_k$.

(iii) Show that the partial Fourier sums $S_{N_k} f(0)$ satisfy a lower bound

$$S_{N_k} f(0) \geq C \log m_k.$$ 

(iv) Find a choice of $m_k$ and $N_k$ so that the Fourier series of $f(x)$ at $x = 0$ diverges.

2. Set

$$Mf(x) = \sup_{\delta > 0} \frac{1}{\mu(B(x, \delta))} \int_{B(x, \delta)} |f(y)| \, dy.$$

with the supremum taken over all balls $B(x, \delta)$, and Set

$$\tilde{M}f(x) = \sup_{\delta > 0} \frac{1}{\mu(B(z, \delta))} \int_{B(z, \delta)} |f(y)| \, dy.$$

with the supremum taken over all balls $B(z, \delta)$ that contain the point $x$.

(i) Show that if $f \in L^1$ and $f \neq 0$ identically then there exist $C, R > 0$ so that $\tilde{M} f(x) \geq C |x|^{-n}$ (here $n$ is the space dimension) for all $x$ with $|x| \geq R$. Hence $m(\{x : \tilde{M}(x) > \alpha\}) \geq C'/\alpha$. Show also that $M(x) \leq \tilde{M}(x) \leq 2^n M(x)$.

(ii) Show that $|f(x)| \leq Mf(x)$ at every Lebesgue point of $f$ if $f \in L^1(\mathbb{R}^n)$.

3. (i) A measurable function $f$ is said to be in weak $L^1$ denoted by $L^1_w$ if $m(\lambda) = \lambda \times |\{x : |f(x)| > \lambda\}|$ is a bounded function of $\lambda \geq 0$. Show that $L^1_w$ contains $L^1$ but is larger than $L^1$.

(ii) Let $f(x) \in C(\mathbb{R})$, $f(x) > 0$ for $0 < x < 1$ and $f(x) = 0$ otherwise. Show
that the function $h_c(x) = \sup_n \{n^c f(nx)\}$ is (i) in $L^1(\mathbb{R})$ if $c \in (0, 1)$, (ii) is in $L^1_w(\mathbb{R})$ but not in $L^1(\mathbb{R})$ if $c = 1$, (ii) not in $L^1_w$ if $c > 1$.

4. (i) Let $g_n = \chi_{[-n,n]}(x)$, compute $h_n = g_n \ast g_1$ and show that $h_n$ is a Fourier transform of a multiple of the function

$$f_n(x) = \frac{\sin x \sin(nx)}{x^2}.$$

(ii) Use (i) to show that the Fourier transform maps $L^1$ into a proper subset of $C_0(\mathbb{R})$ and not onto $C_0(\mathbb{R})$.

(iii) Show that the image of the Fourier transform is dense in $C_0(\mathbb{R})$.

5. (i) Let $f \in C(S^1)$ have a modulus of continuity $\omega(\delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|$. Show that $|\hat{f}(n)| \leq C\omega(1/2n)$.

(ii) Assume that $f$ is absolutely continuous, show that $\hat{f}(n) = o(1/n)$ as $n \to +\infty$.

(iii) Show that $f \in L^1(S^1)$ is equal to an analytic function a.e. on $S^1$ if and only if there exist $c > 0$ and $A > 0$ so that $|\hat{f}(n)| \leq Ae^{-cn}$. 